بسم الله الرحمن الرحيم

Essence of

Critical Phenomena:

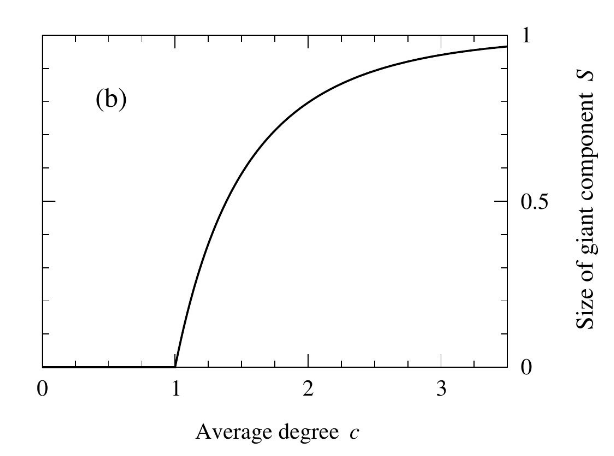
Phase Transitions & RG

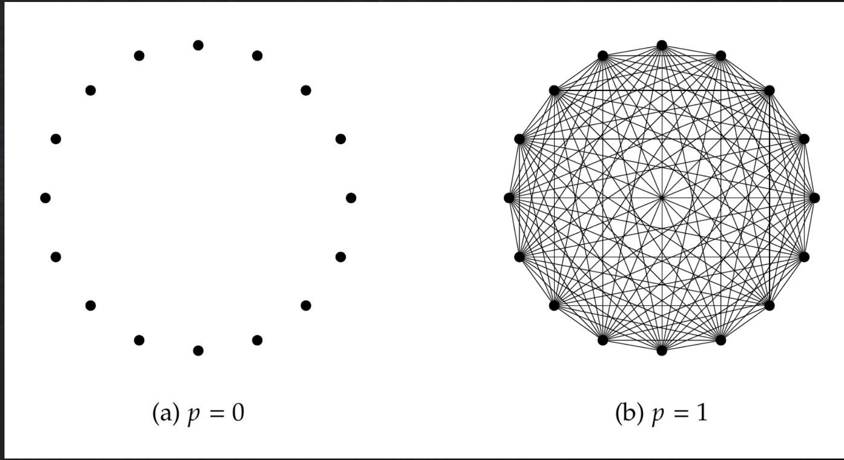
Abbas K. Rizi

CX Group, CS Dept., Aalto University

abbas.sitpor.org

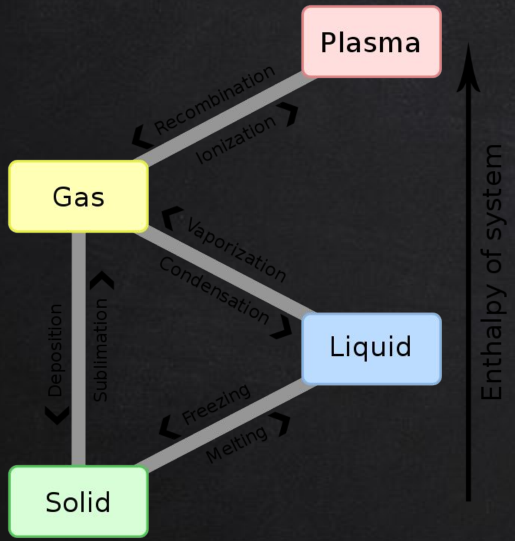
Emergence of giant component in an ER Network





Networks, 2nd edition. Mark Newman, Oxford University Press (2018). © Mark Newman.
DOI: 10.1093/oso/9780198805090.001.0001

Phase Transitions



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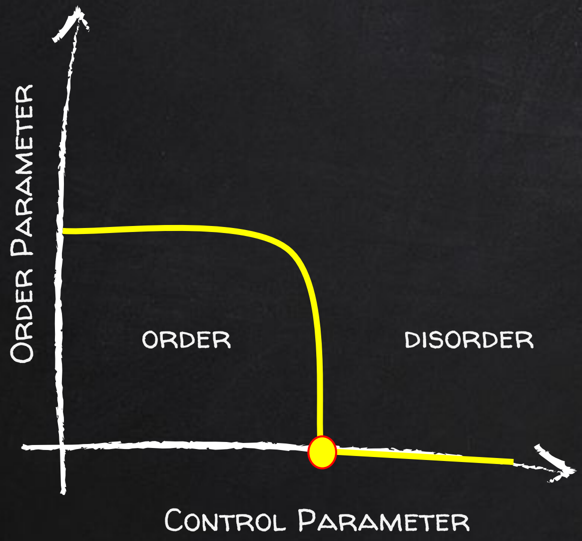
Phase Transitions

Phase Transitions



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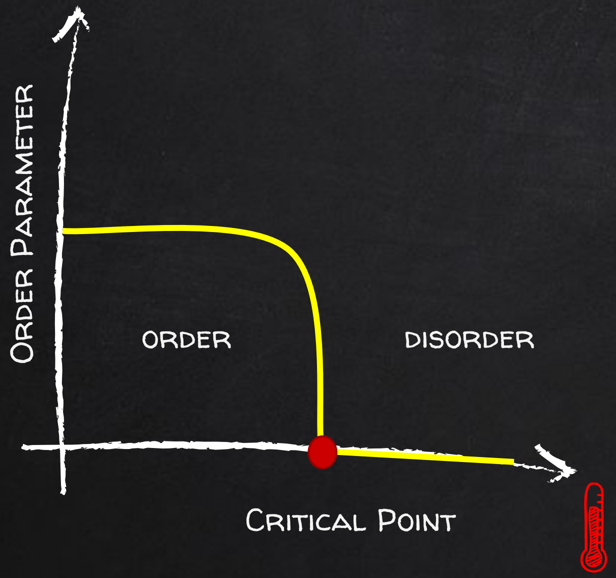
Phase Transitions



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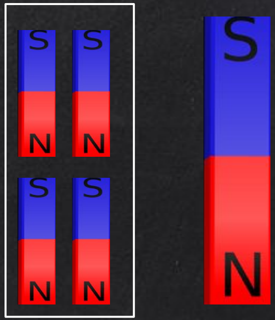
Phase Transitions

Order Parameters:

- Magnetization: $\langle \sigma \rangle$
- Density of Particles: ρ



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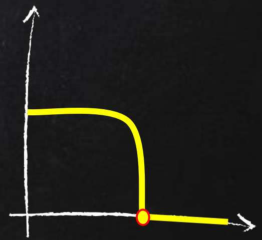
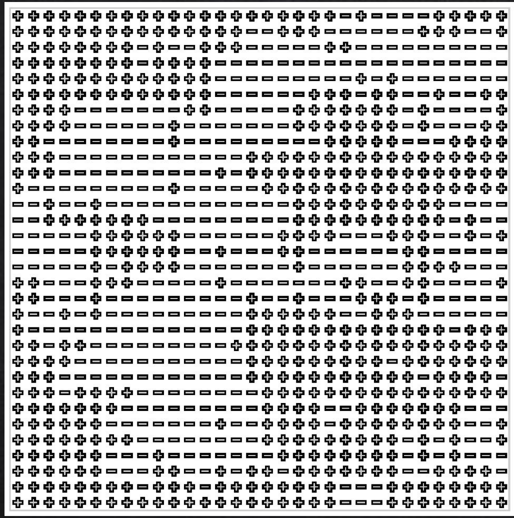
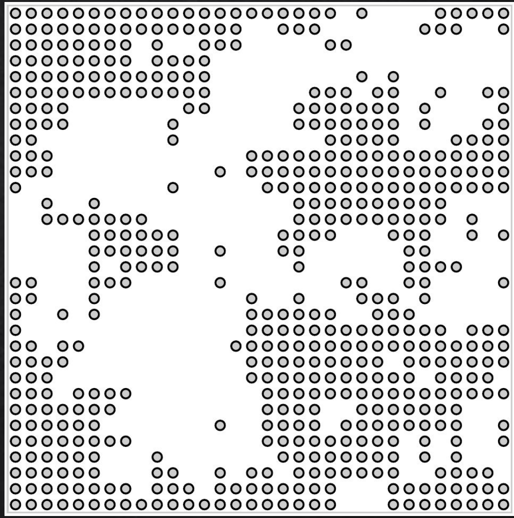
Ising Model & Phase Transition



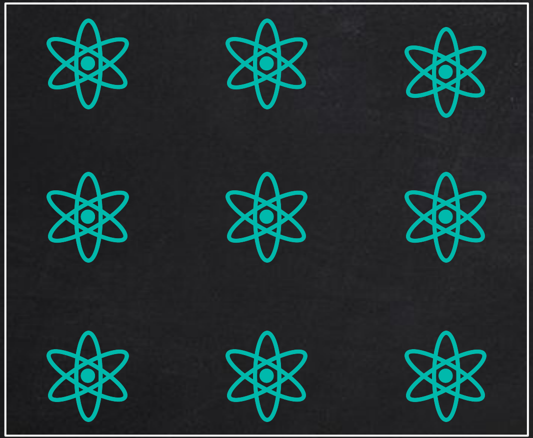


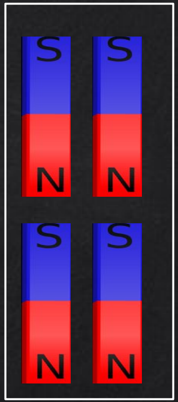


Configurations of the Ising model on a two-dimensional square lattice considered as a magnet (left) and as a lattice gas (right).

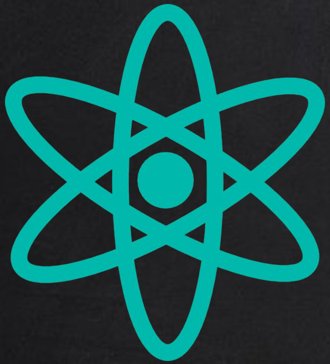
Ising Model

Ising Model



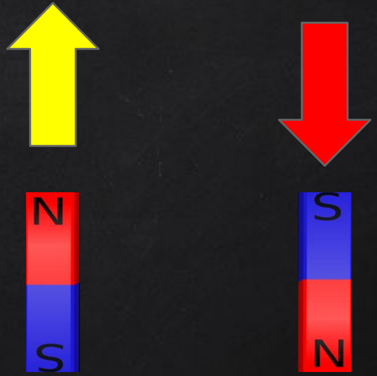


Ising Model

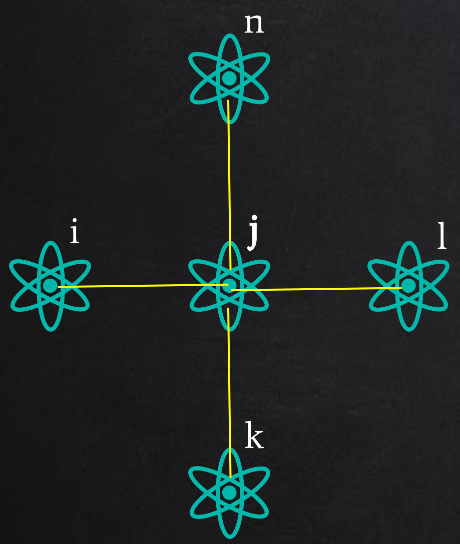


i

σ_i : +1, −1

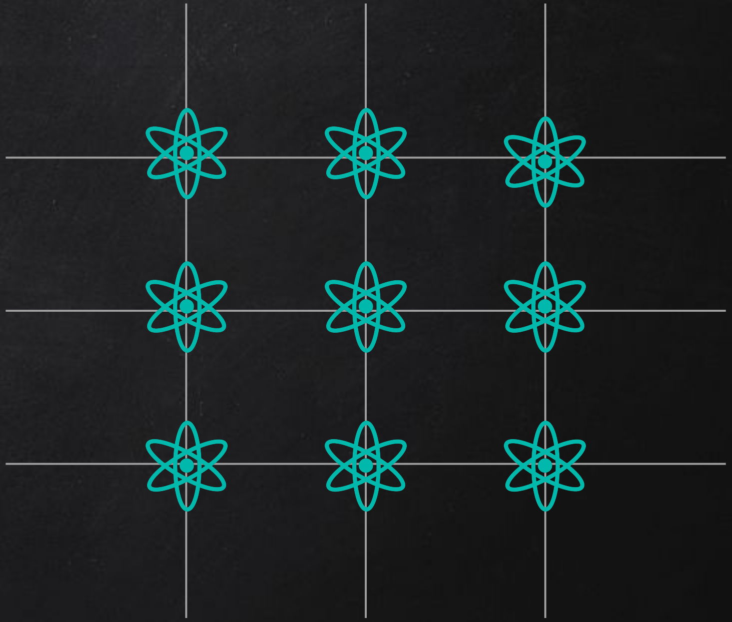


Ising Model



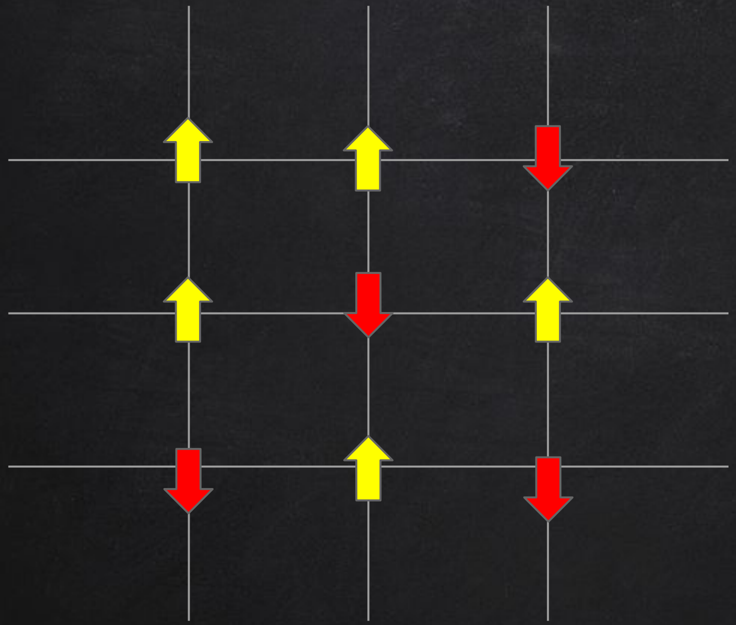
$J_{ij} = 1$

$J_{mj} = 0$

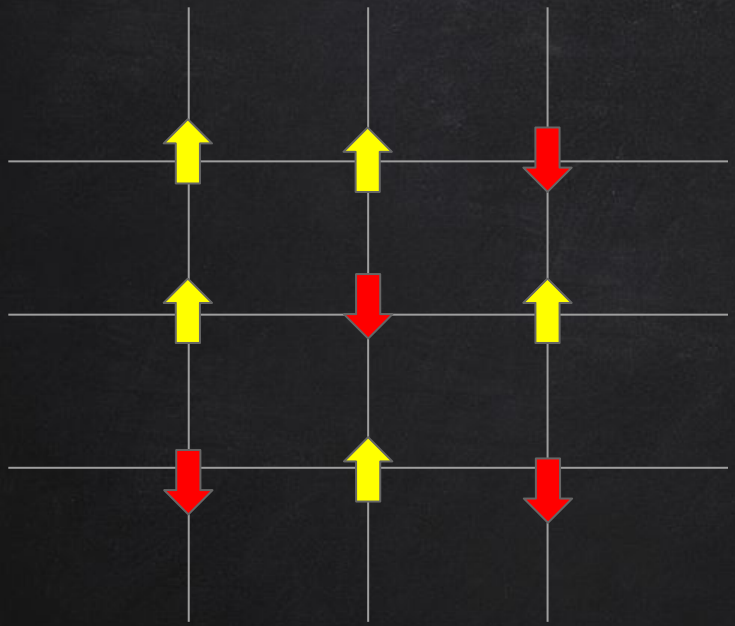


Network (Lattice)

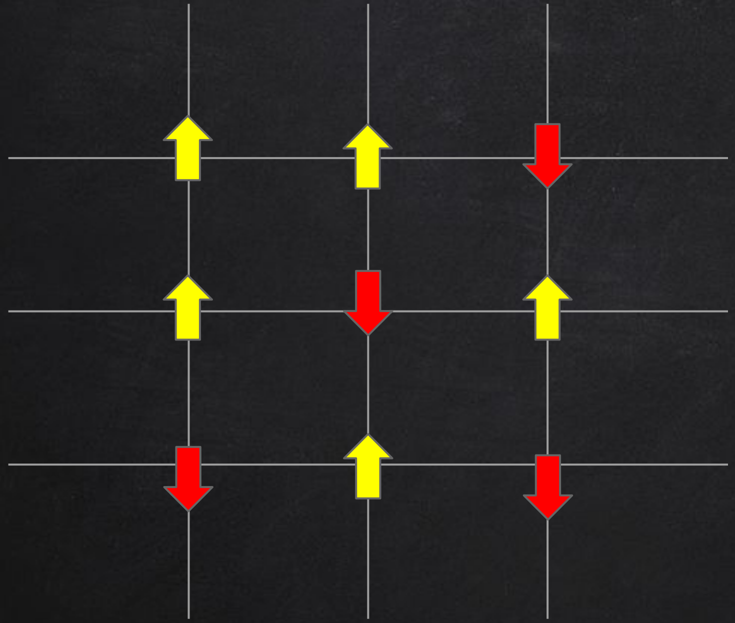
Ising Model



Ising Model

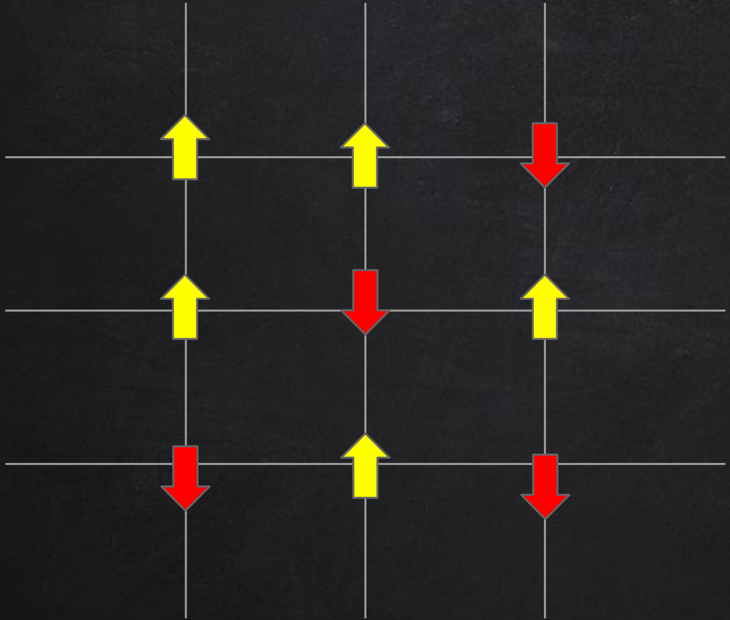


Ising Model



$$\{\sigma_x\}$$

Ising Model

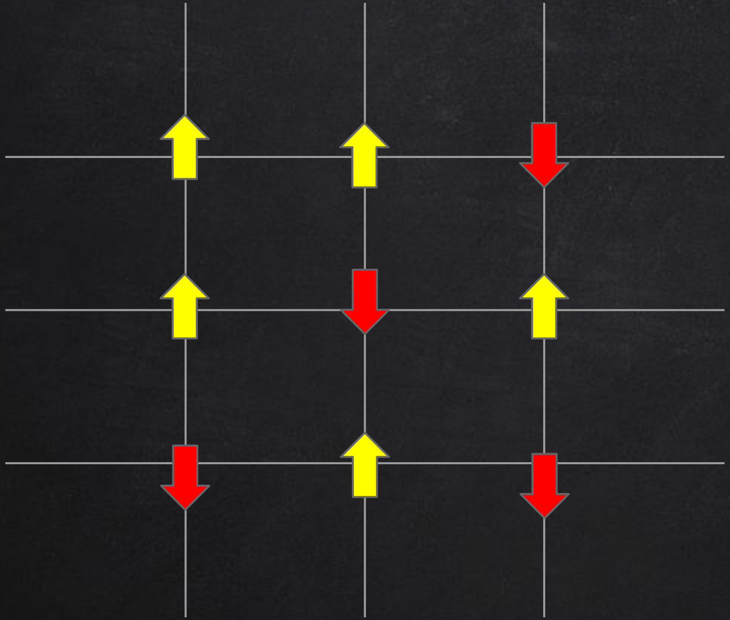


$$\{\sigma_x\}$$

$$P(\{\sigma_x\}) = \frac{e^{\beta \sum_{i>j} J_{ij}\sigma_i\sigma_j}}{Z(\beta)}$$

Ising Model

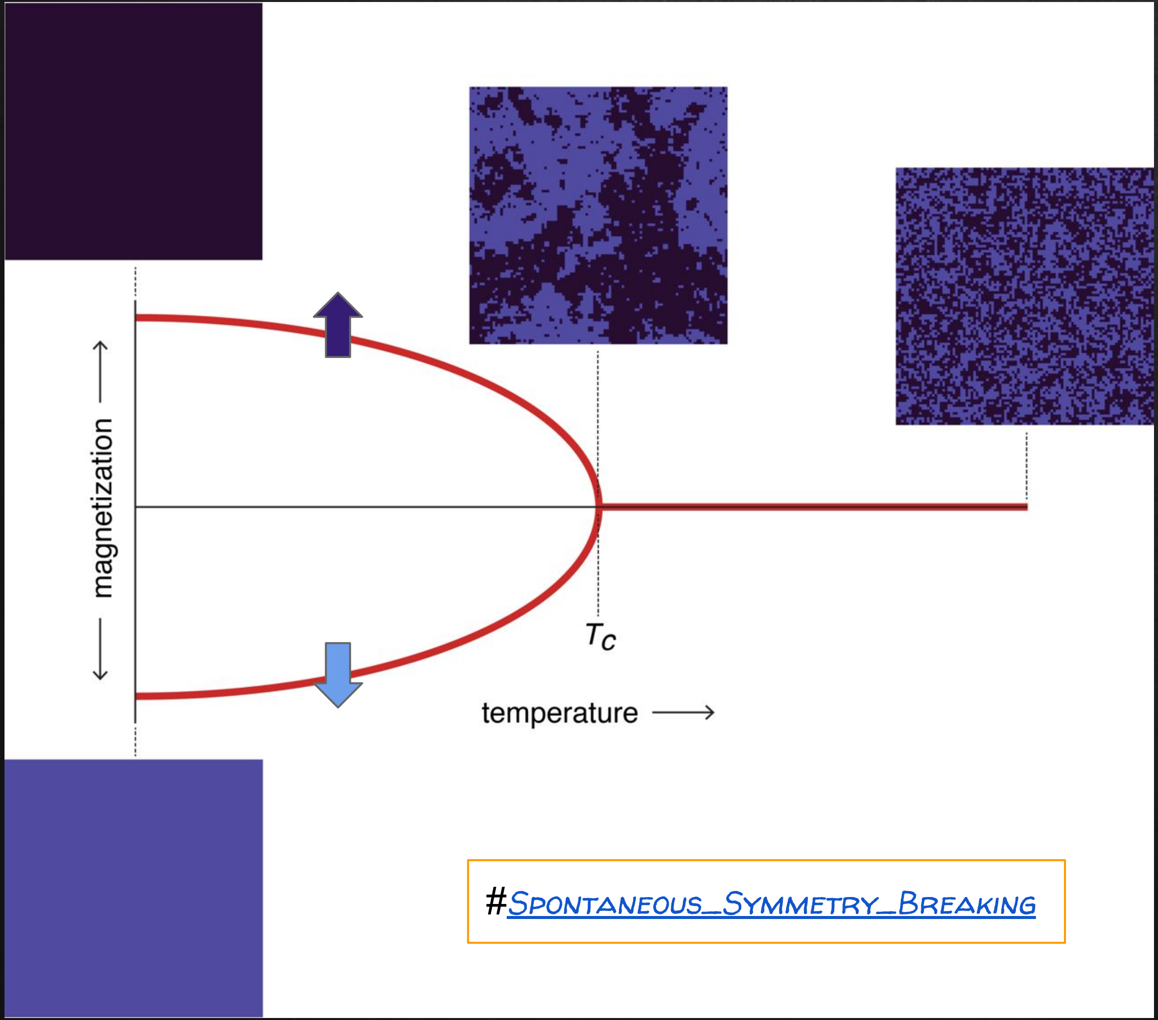
$\frac{1}{\beta}$



$\{\sigma_x\}$

$$P(\{\sigma_x\}) = \frac{e^{\beta \sum_{i>j} J_{ij}\sigma_i\sigma_j}}{Z(\beta)}$$

Partition Function

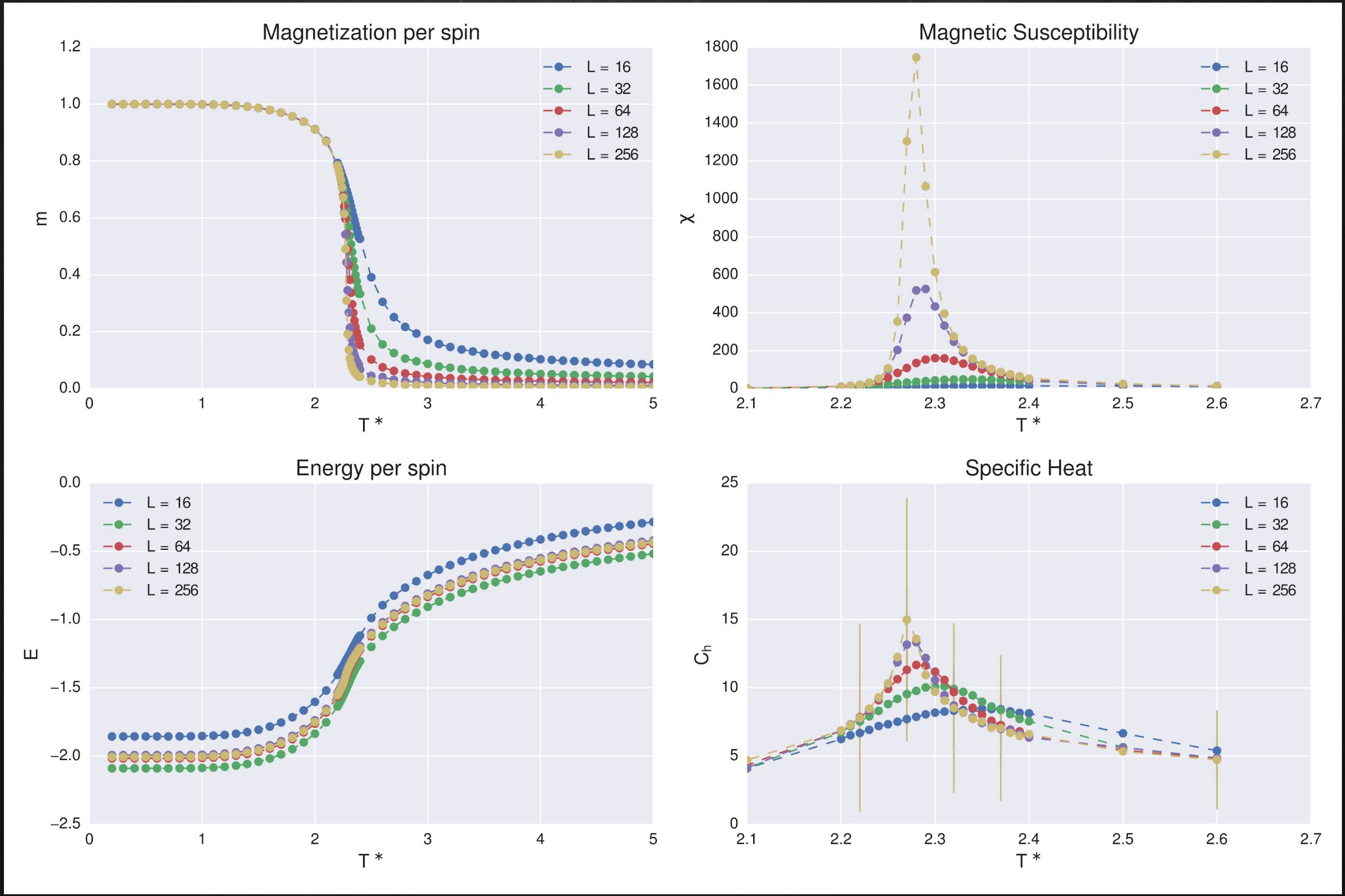


#Spontaneous_Symmetry_Breaking

$$P(\{\sigma_x\}) = \frac{e^{\beta \sum_{i>j} J_{ij}\sigma_i\sigma_j}}{Z(\beta)}$$

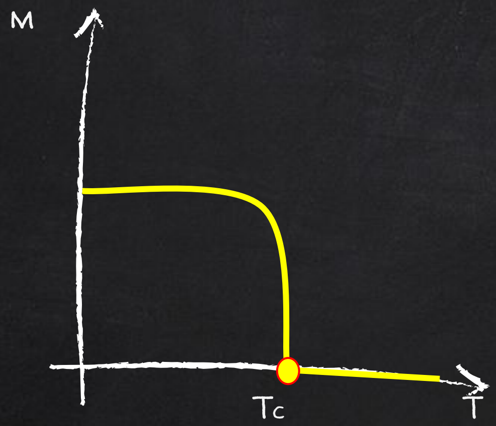
$$m = < \sigma_i >$$

bit-player.org



github.com/VictorSeven/IsingModel

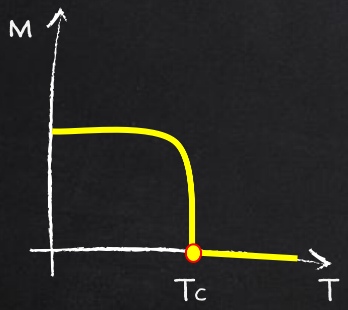
Scalings & Universalities



$$m(T) \sim \tau^{\beta}$$

$$\tau = \frac{T_c - T}{T_c}$$

Scalings & Universalities



$$\tau = \frac{T_c - T}{T_c}$$

$$m(T) \sim \tau^{\beta}$$

$$\chi(T) \sim |\tau|^{-\gamma}$$

$$C(T) \sim |\tau|^{-\alpha}$$

$$\xi(T) \sim |\tau|^{-\nu}$$

$$m(T_c, h) \sim |h|^{1/\delta}$$

$$G(r; T_c) \sim r^{-d+2-\eta}$$

Hyperscaling Relations

$$m(T) \sim \tau^{\beta}$$
$$\chi(T) \sim |\tau|^{-\gamma}$$
$$C(T) \sim |\tau|^{-\alpha}$$
$$\xi(T) \sim |\tau|^{-\nu}$$
$$m(T_c, h) \sim |h|^{1/\delta}$$
$$G(r; T_c) \sim r^{-d+2-\eta}$$

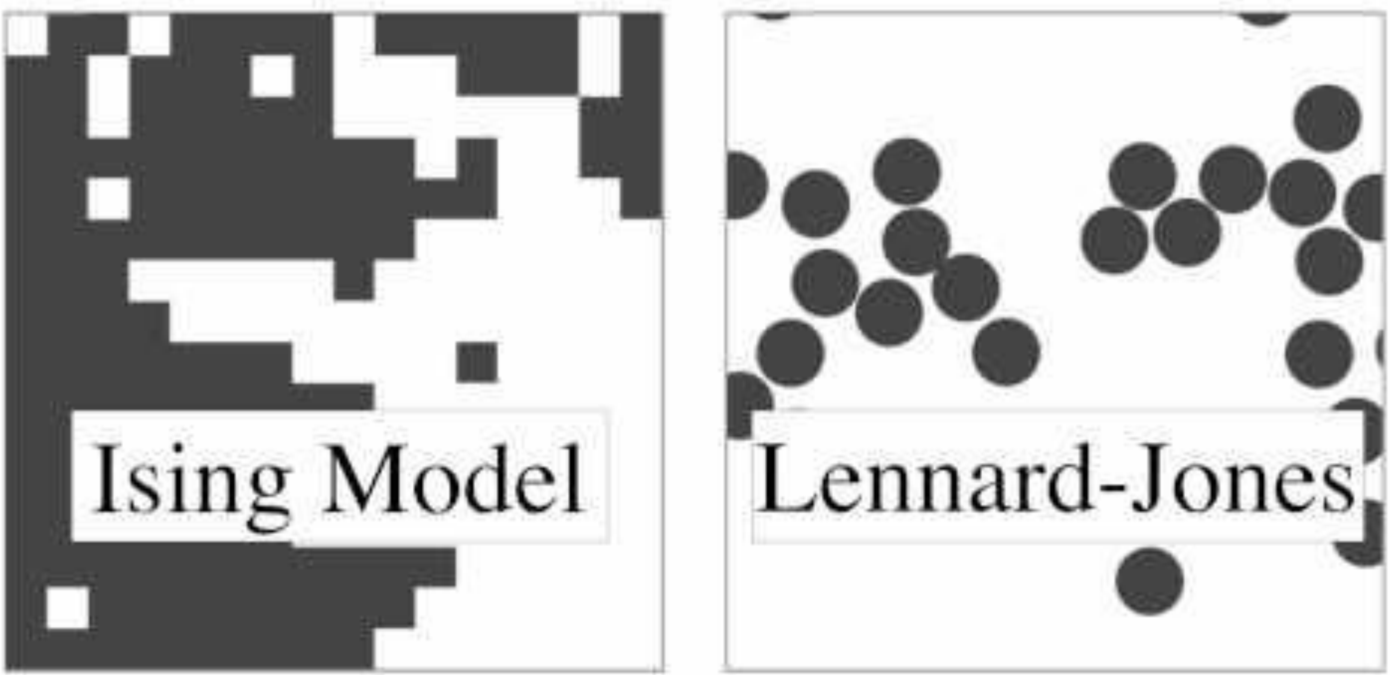
$$\alpha + 2\beta + \gamma = 2$$ Rushbrooke

$$\gamma = \beta(\delta - 1)$$ Widom

$$\gamma = (2 - \eta)\nu$$ Fisher

$$2 - \alpha = \nu d$$ Josephson

https://youtu.be/Kp4UvhUsBAU



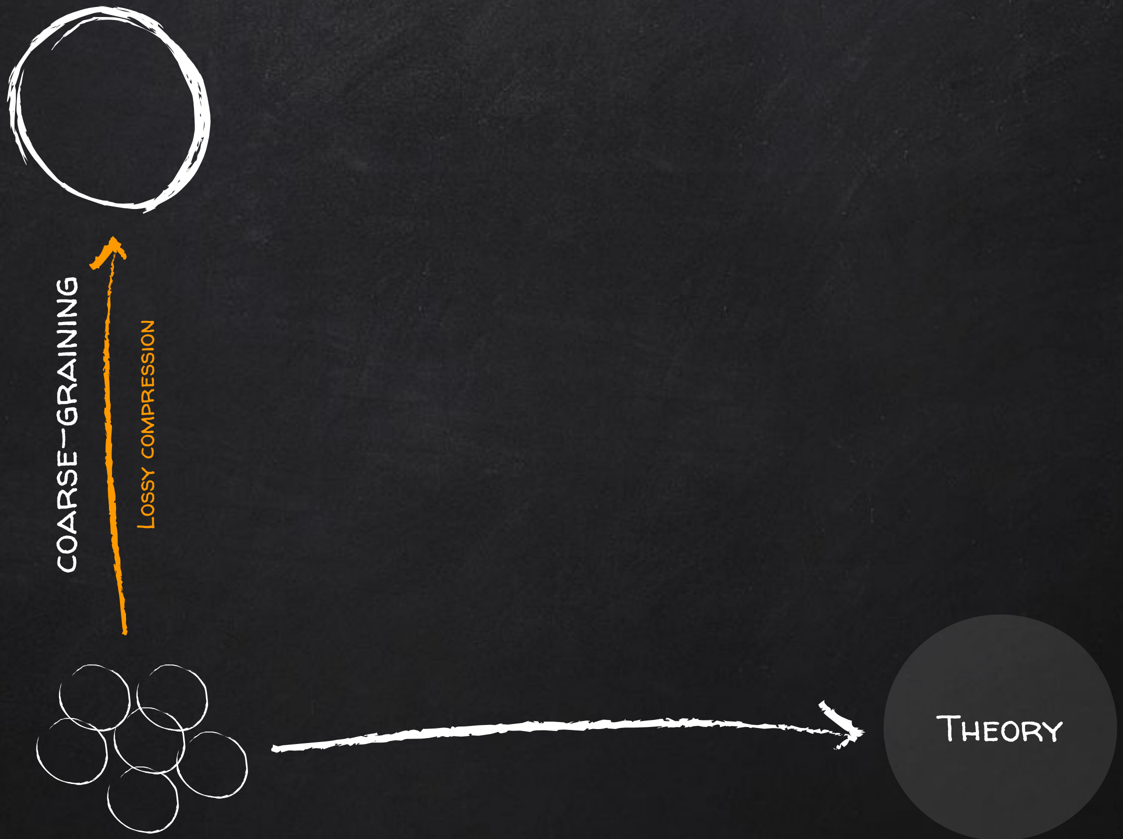
Renormalization

Introduction to Renormalization

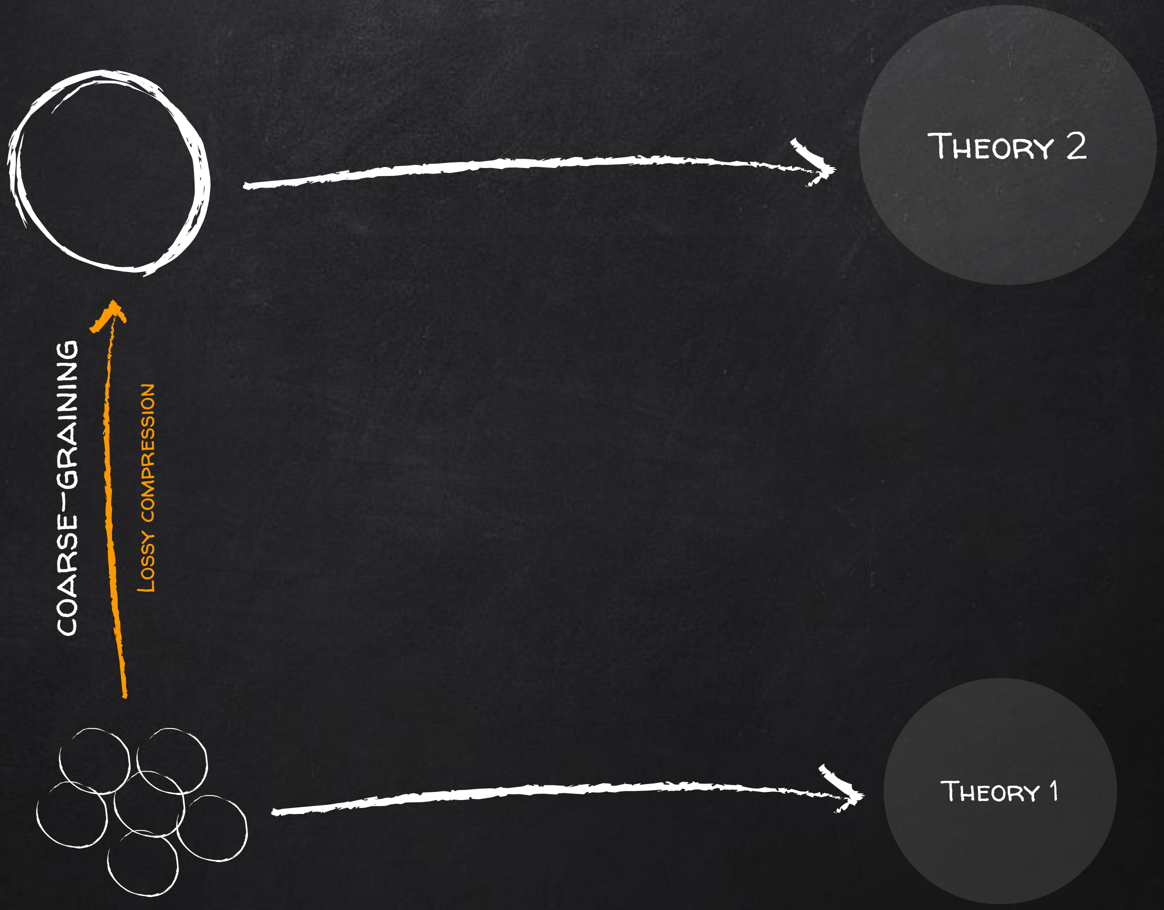
Simon DeDeo, Ph.D. in Astrophysics

- Assistant Professor, Laboratory for Social Minds, Carnegie Mellon University
- External Faculty at the Santa Fe Institute
- http://bit.ly/SFIrenorm

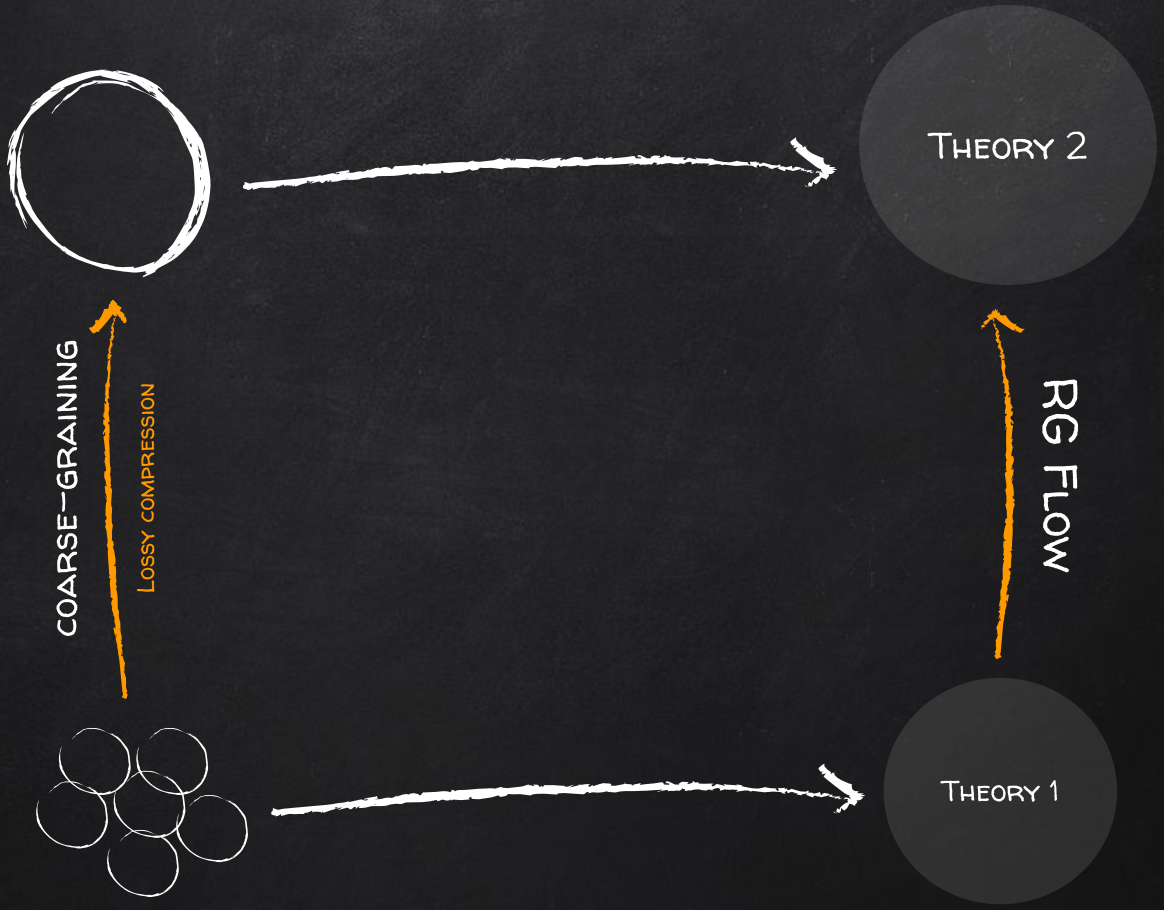
Renormalization



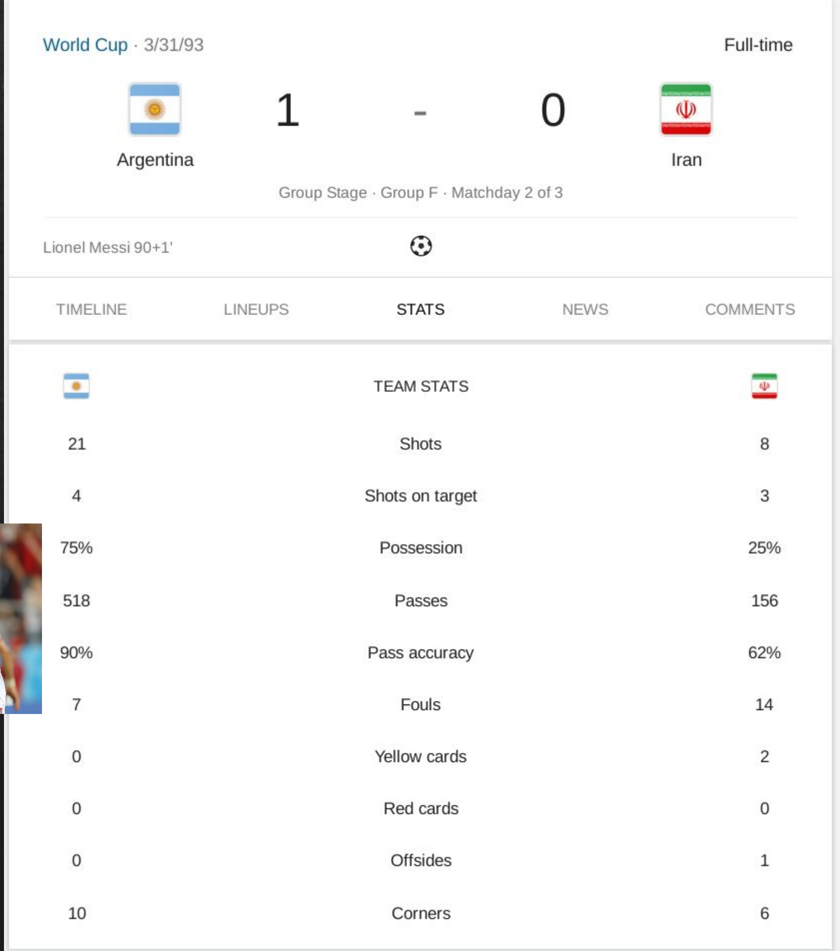
COARSE-GRAINING
Lossy compression
Theory



Theory 2
Coarse-graining
Lossy compression
Theory 1



Theory 2
coarse-graining
Lossy compression
RG Flow
Theory 1


World Cup · 3/31/93 Full-time

Argentina 1 - 0 Iran

Group Stage · Group F · Matchday 2 of 3

Lionel Messi 90+1'

TIMELINE LINEUPS STATS NEWS COMMENTS

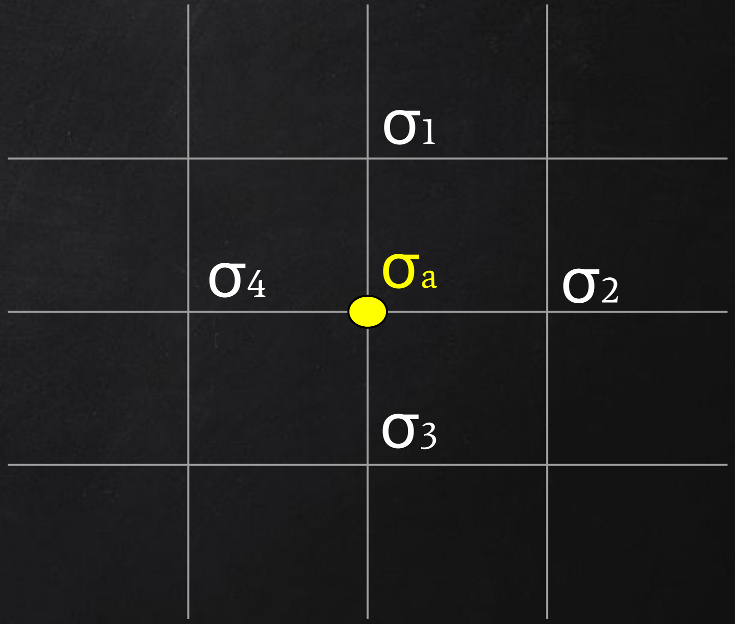
Argentina	TEAM STATS	Iran
21	Shots	8
4	Shots on target	3
75%	Possession	25%
518	Passes	156
90%	Pass accuracy	62%
7	Fouls	14
0	Yellow cards	2
0	Red cards	0
0	Offsides	1
10	Corners	6

Real Space Renormalization Group (RSRG)

Renormalizing the Ising Model

$$P(\{\sigma_x\}) = \frac{e^{\beta \sum_{i>j} J_{ij}\sigma_i\sigma_j}}{Z(\beta)}$$

Decimation



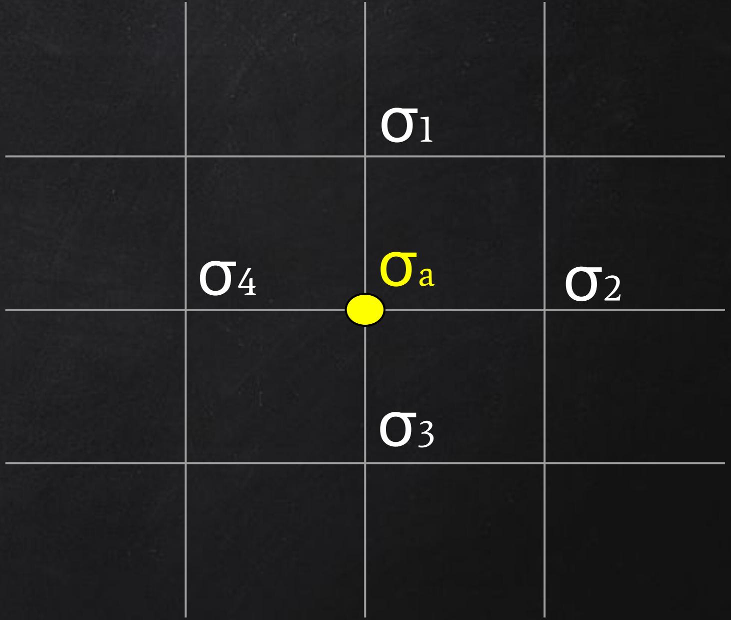
$$P(\{\sigma_x\}) = \frac{e^{\beta \sum_{i>j} J_{ij}\sigma_i\sigma_j}}{Z(\beta)}$$

$$P(\{\sigma_x\}) = \frac{B}{Z} e^{\beta(\sigma_a\sigma_1+\sigma_a\sigma_2+\sigma_a\sigma_3+\sigma_a\sigma_4)}$$

$$B = e^{\sum_{i,j\neq a} \beta J_{ij}\sigma_i\sigma_j}$$

Renormalizing the Ising Model

$$\sum_{\sigma_a=\pm 1} P(\{\sigma_i\})$$



#trace_out

$$\sum_{\sigma_a=\pm 1} P(\{\sigma_i\})$$

$$\sum_{\sigma_a=\pm 1} P(\{\sigma_i\}) = \frac{B}{Z}\left(e^{\beta(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} + e^{-\beta(\sigma_1+\sigma_2+\sigma_3+\sigma_4)}\right)$$

$$\sum_{\sigma_a=\pm 1} P(\{\sigma_i\}) = \frac{B}{Z}(e^{\beta(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} + e^{-\beta(\sigma_1+\sigma_2+\sigma_3+\sigma_4)})$$

$$= \frac{B}{Z}\Big(2\cosh\beta(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)\Big)$$

$$\sum_{\sigma_a=\pm 1} P(\{\sigma_i\}) = \frac{B}{Z}\left(e^{\beta(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} + e^{-\beta(\sigma_1+\sigma_2+\sigma_3+\sigma_4)}\right)$$

$$= \frac{B}{Z'}\left(e^{\ln\cosh\beta(\sigma_1+\sigma_2+\sigma_3+\sigma_4)}\right)$$

$$P(\{\sigma_i\}) = \frac{e^{\beta \sum_{i>j} J_{ij}\sigma_i\sigma_j}}{Z(\beta)}$$

COARSE-GRAINING

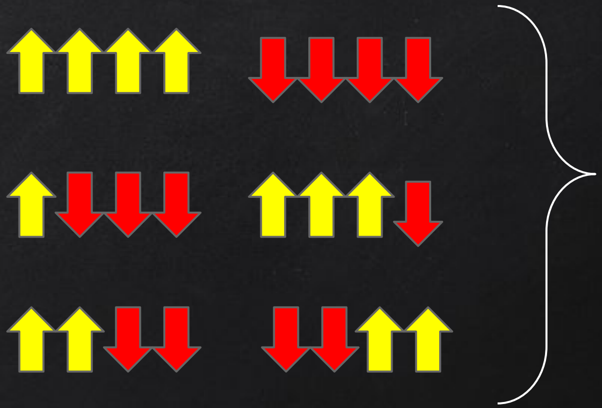
$$\sum_{\sigma_a=\pm 1} P(\{\sigma_i\}) = \frac{B}{Z'}\left(e^{\ln\cosh\beta(\sigma_1+\sigma_2+\sigma_3+\sigma_4)}\right)$$

$$\sum_{\sigma_a=\pm 1} P(\{\sigma_i\}) = \frac{B}{Z'}\left(e^{\ln\cosh\beta(\sigma_1+\sigma_2+\sigma_3+\sigma_4)}\right)$$

$$ln\cosh(4\beta)$$

$$ln\cosh(2\beta)$$

$$ln\cosh(0) = 0$$



Inducing Quartets and Commutation Failure

$$\sum_{\sigma_a=\pm 1} P(\{\sigma_i\}) = \frac{B}{Z'}\left(e^{\ln\cosh\beta(\sigma_1+\sigma_2+\sigma_3+\sigma_4)}\right)$$

$$= \frac{B}{Z'}\left(e^{S_2(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4)+S_4(\sigma_1\sigma_2\sigma_3\sigma_4)}\right)$$

$$\sum_{\sigma_a=\pm 1} P(\{\sigma_i\}) = \frac{B}{Z'}\left(e^{\ln\cosh\beta(\sigma_1+\sigma_2+\sigma_3+\sigma_4)}\right)$$

$$= \frac{B}{Z'}\left(e^{S_2(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4)+S_4(\sigma_1\sigma_2\sigma_3\sigma_4)}\right)$$

$$S_2 = \frac{1}{8} ln \cosh 4\beta$$

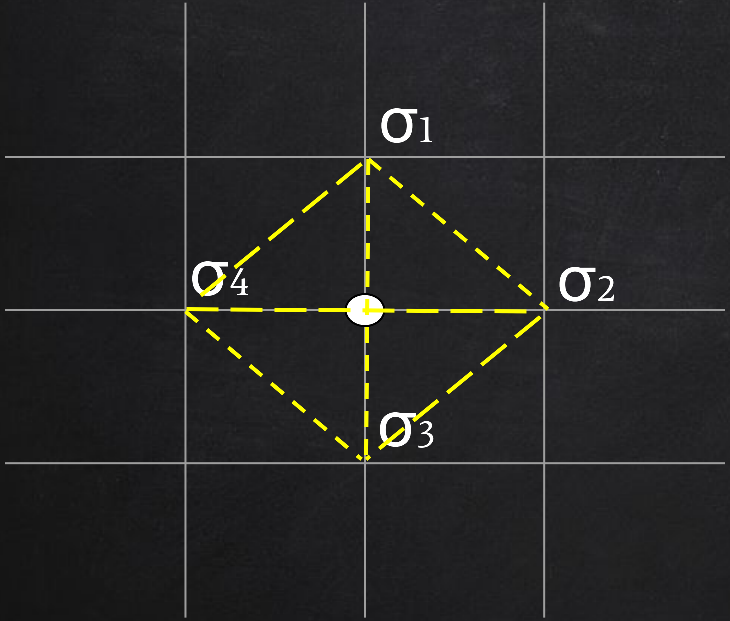
$$S_4 = \frac{1}{8} ln \cosh 4\beta - \frac{1}{2} ln \cosh 2\beta$$

$$\sum_{\sigma_a=\pm 1} P(\{\sigma_i\})$$

$$= \frac{B}{Z'}\left(e^{S_2(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4)+S_4(\sigma_1\sigma_2\sigma_3\sigma_4)}\right)$$

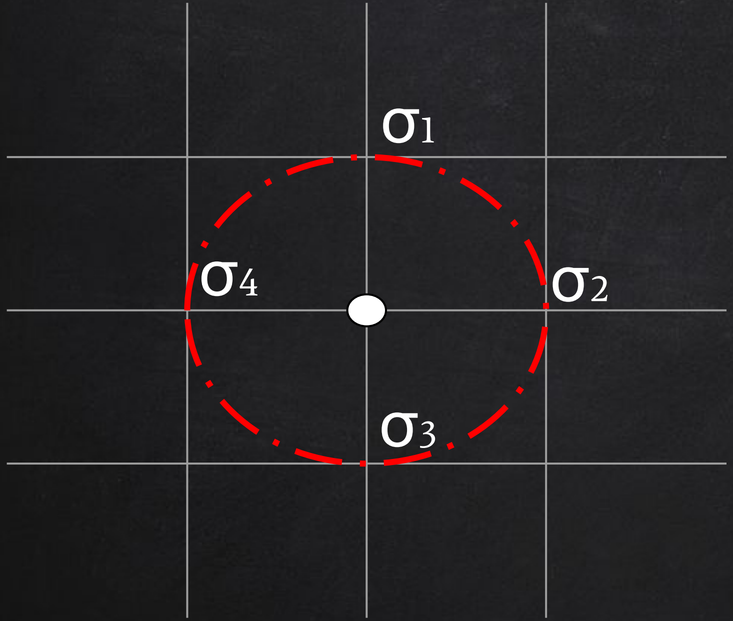
$$S_2 = \frac{1}{8} ln \cosh 4\beta$$

$$S_4 = \frac{1}{8} ln \cosh 4\beta - \frac{1}{2} ln \cosh 2\beta$$



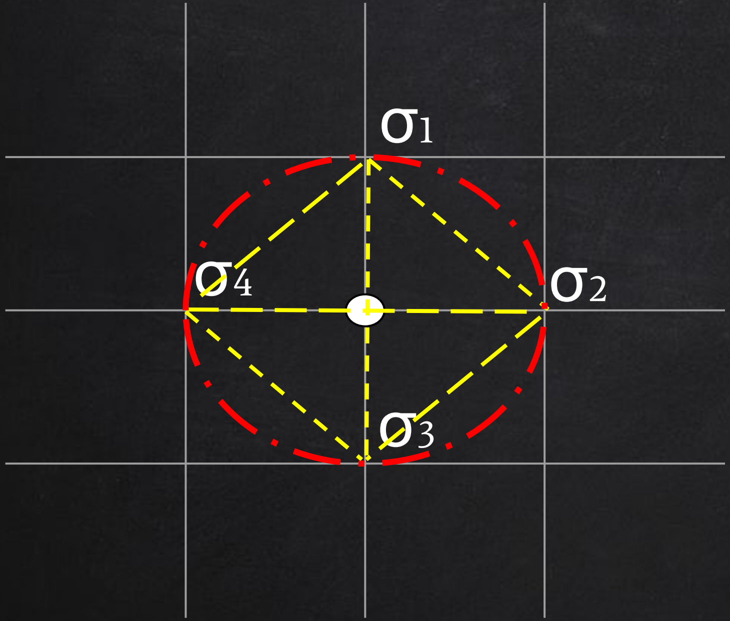
$$S_2 = \frac{1}{8} ln \cosh 4\beta$$

$$= \frac{B}{Z'}\left(e^{S_2(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4)+S_4(\sigma_1\sigma_2\sigma_3\sigma_4)}\right)$$



$$S_4 = \frac{1}{8} ln \cosh 4\beta - \frac{1}{2} ln \cosh 2\beta$$

$$= \frac{B}{Z'} \left(e^{S_2(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4)+S_4(\sigma_1\sigma_2\sigma_3\sigma_4)} \right)$$

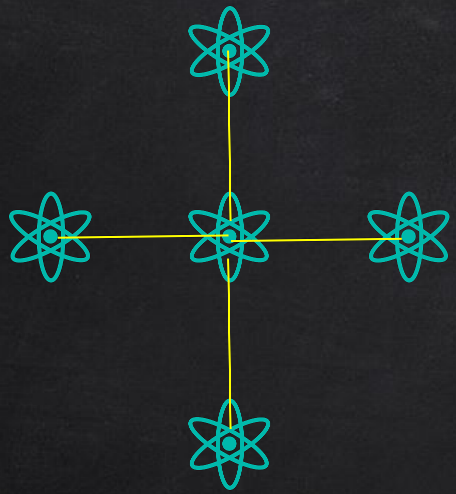


$$S_2 = \frac{1}{8} ln \cosh 4\beta$$

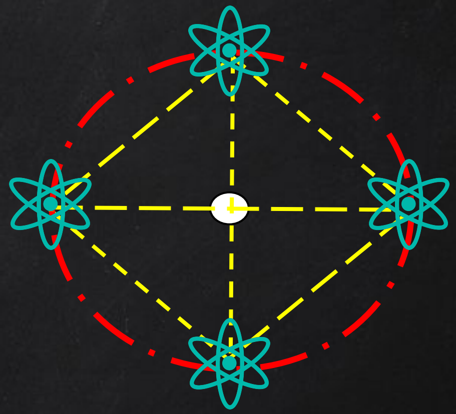
$$S_4 = \frac{1}{8} ln \cosh 4\beta - \frac{1}{2} ln \cosh 2\beta$$

$$= \frac{B}{Z'} \left(e^{S_2(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4)+S_4(\sigma_1\sigma_2\sigma_3\sigma_4)} \right)$$

Changing the network structure



Network J



Network J'

Renormalizing the Ising Model

$$P(\{\sigma_x\}) = \frac{e^{\beta \sum_{i>j} J_{ij}\sigma_i\sigma_j}}{Z(\beta)}$$

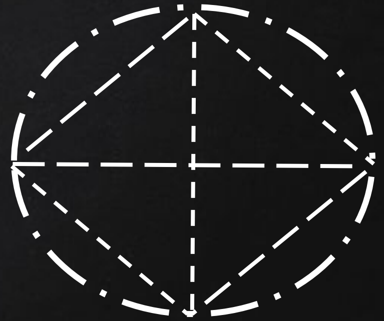
$$P'(\{\sigma_x\}) = \frac{e^{\beta(J'_{ij}\sigma_i\sigma_j + K_{ijkl}\sigma_i\sigma_j\sigma_k\sigma_l)}}{Z'(\beta)}$$

Taking out of the Model Class

#synergy

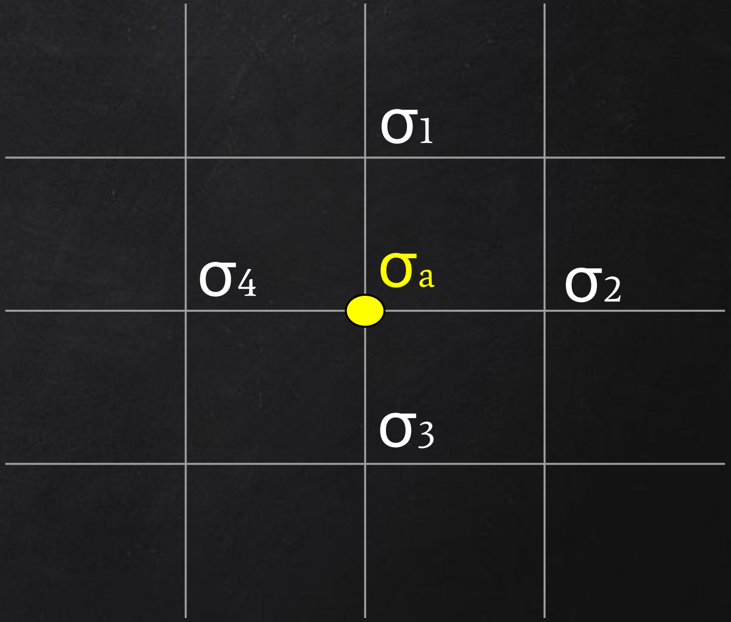
#emergence

$$P'(\{\sigma_x\}) = \frac{e^{\beta(J'_{ij}\sigma_i\sigma_j + \boxed{K_{ijkl}}\sigma_i\sigma_j\sigma_k\sigma_l)}}{Z'(\beta)}$$



Coarse-graining: a single site

$$\sum_{\sigma_a=\pm 1} P(\{\sigma_i\})$$

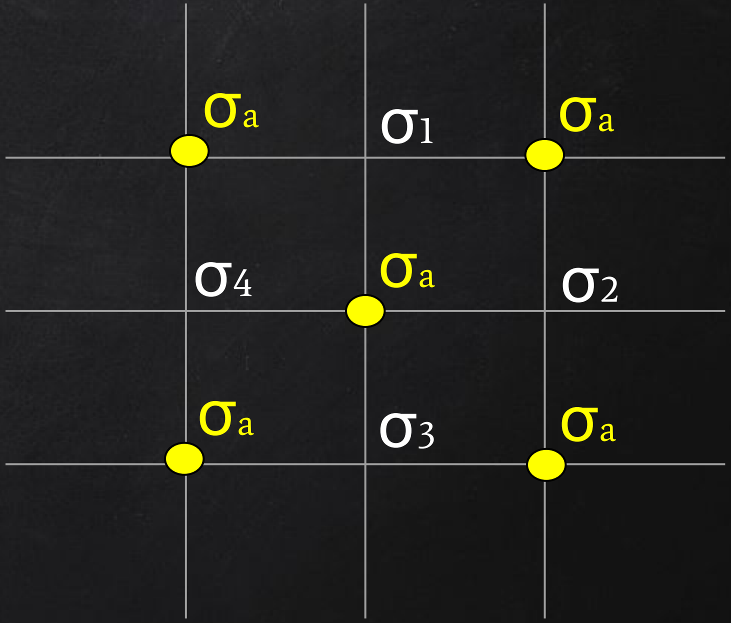


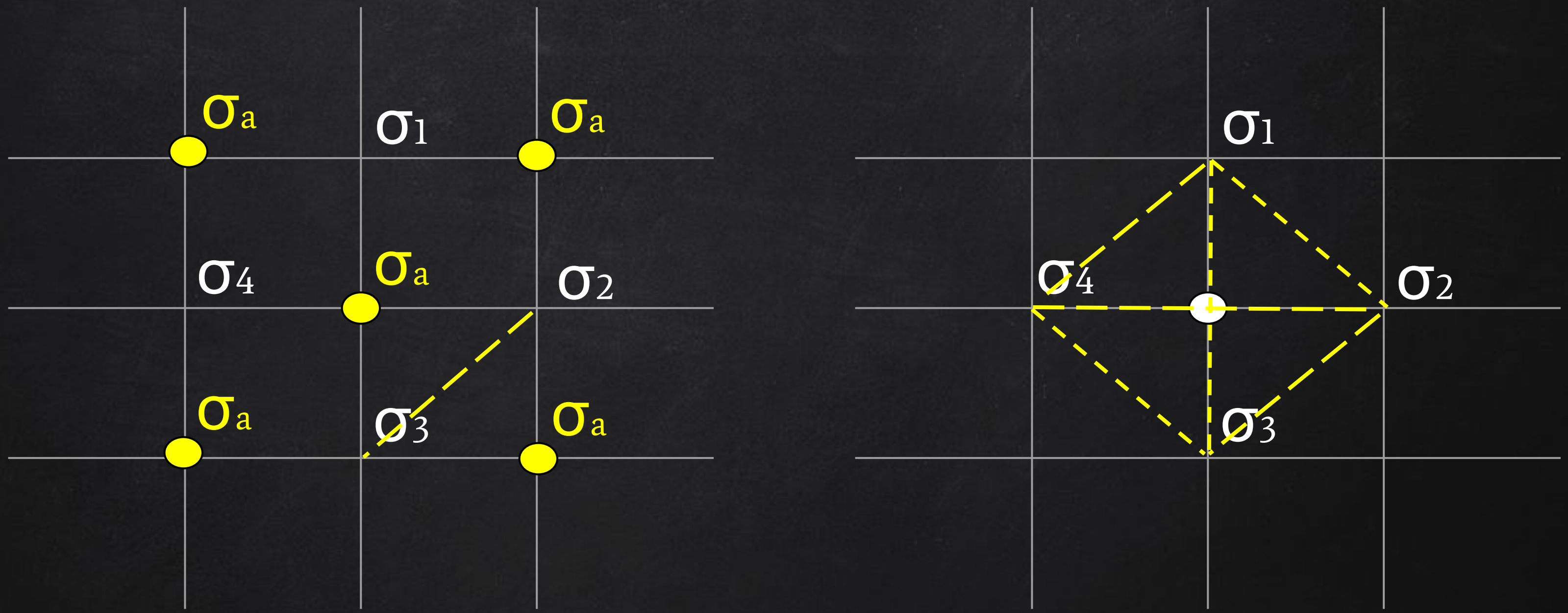
#trace_out

Coarse-graining: Whole Lattice

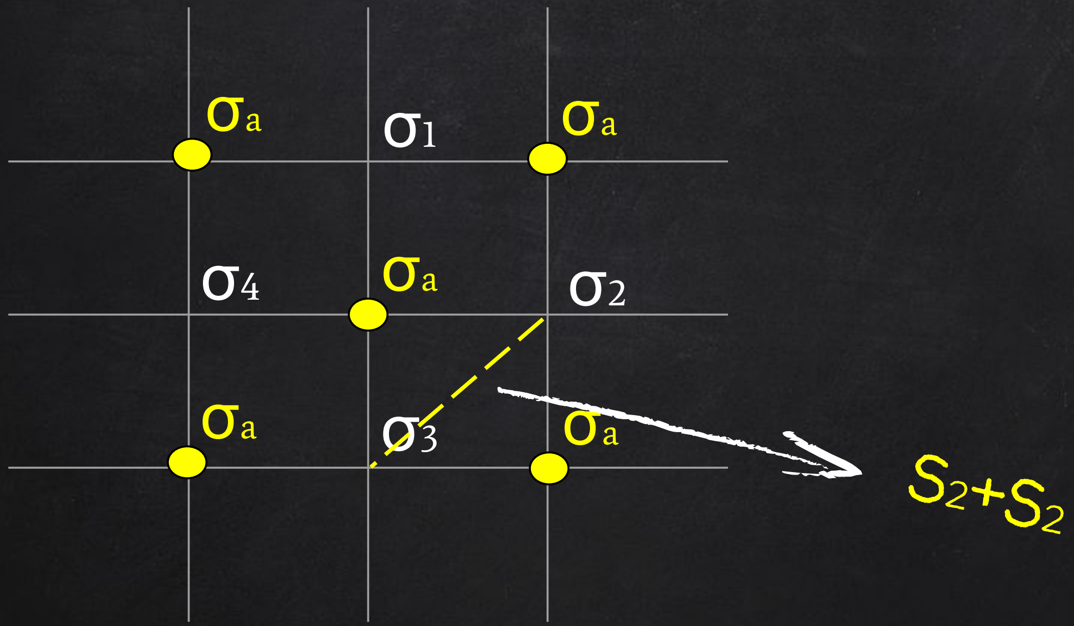
$$\sum_{\sigma_a=\pm 1} P(\{\sigma_i\})$$

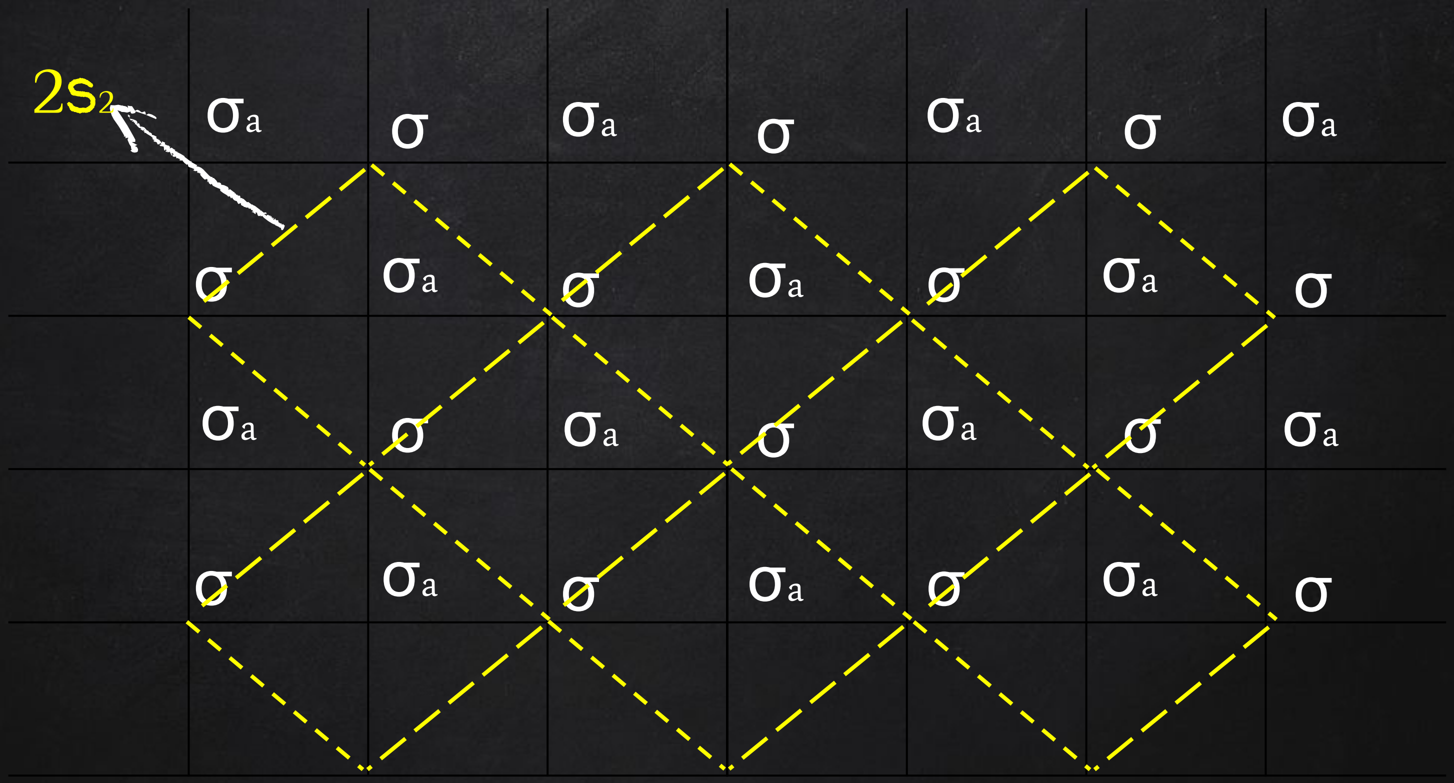
#trace_out





σa
σ1
σa
σ4
σa
σ2
σa
σ3
σa
σ1
σ4
σ2
σ3


σa
σ1
σa
σ4
σa
σ2
σa
σ3
σa
S2+S2


2s2
σa
σ
σa
σ
σa
σ
σa
σ
σa
σ
σa
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σa
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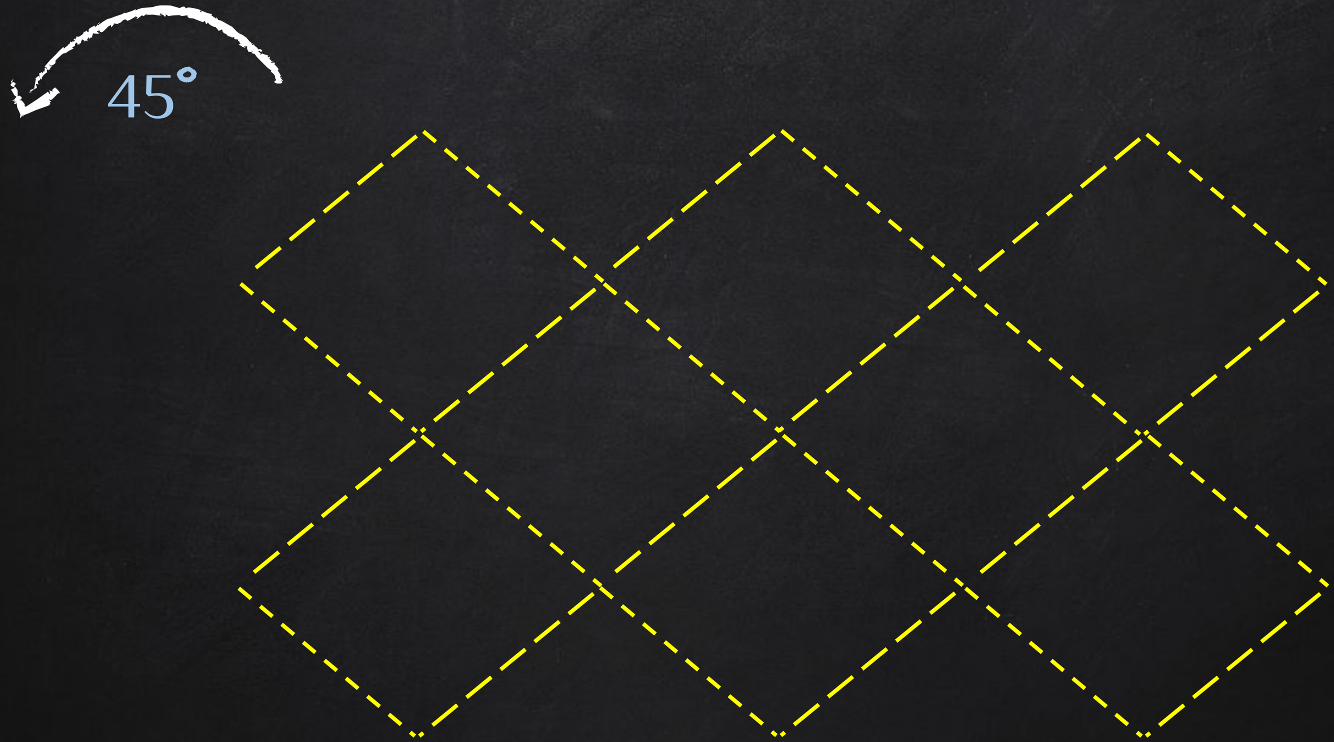
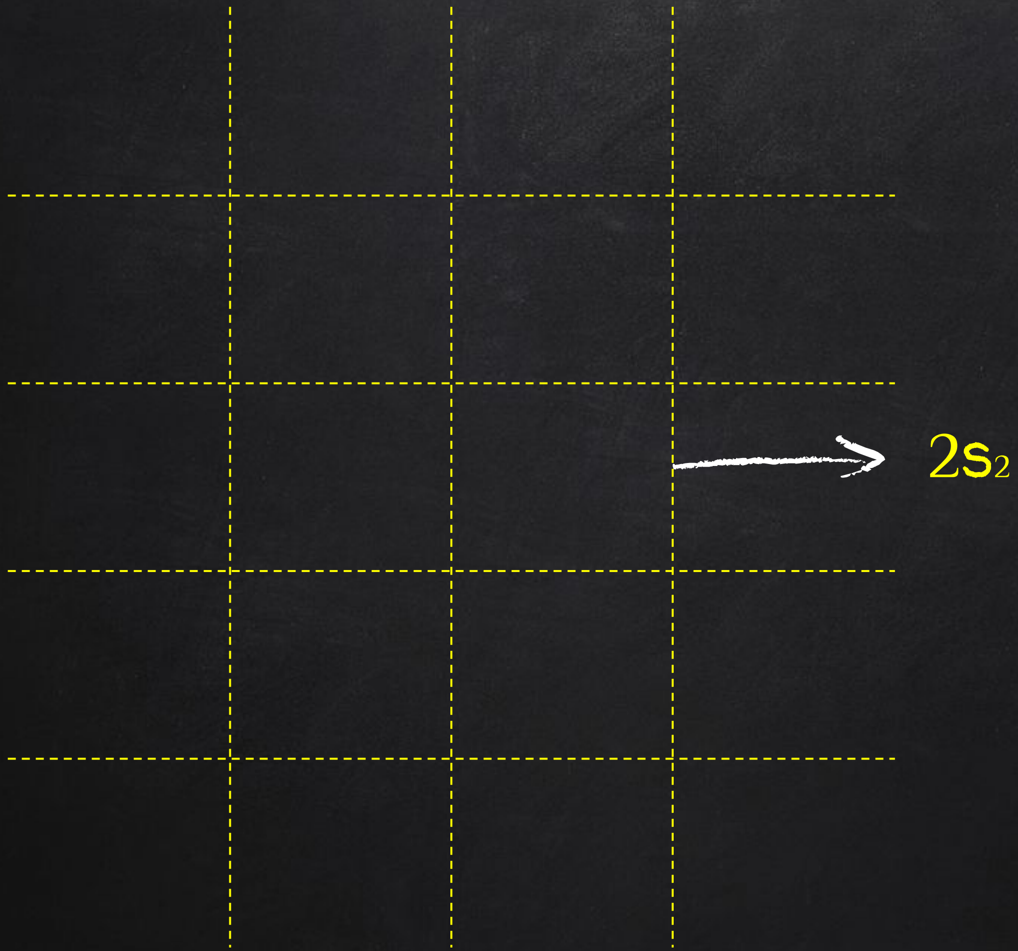
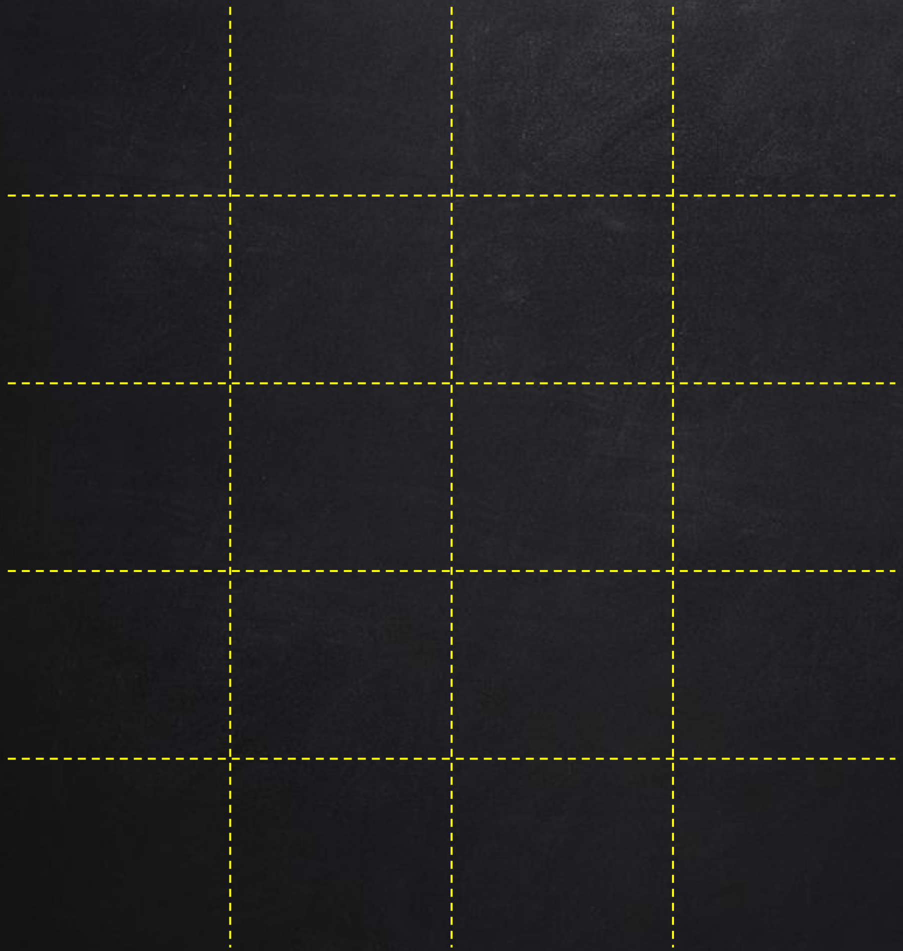

45°



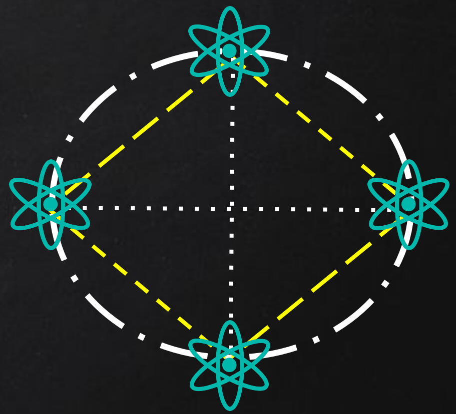
Square Lattice!

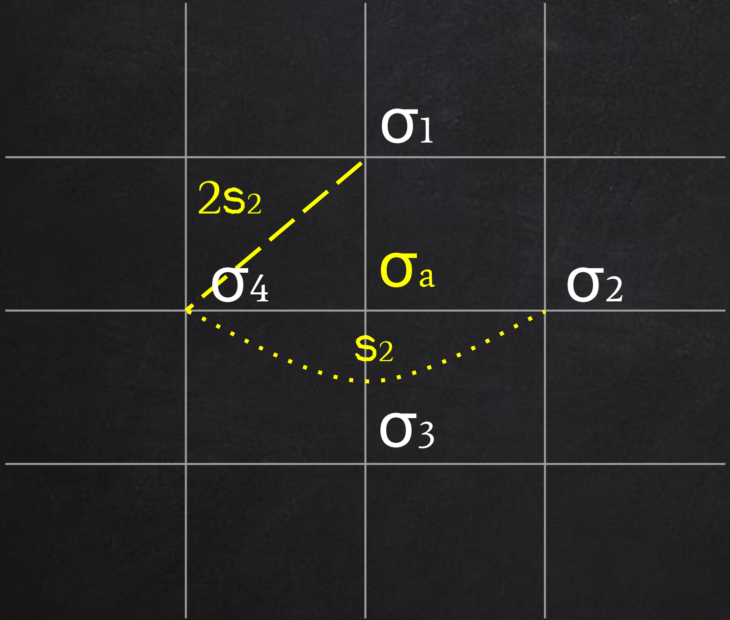
1st NN



Square Lattice!

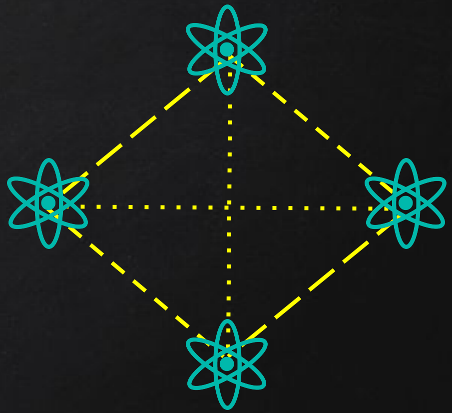
1st NN





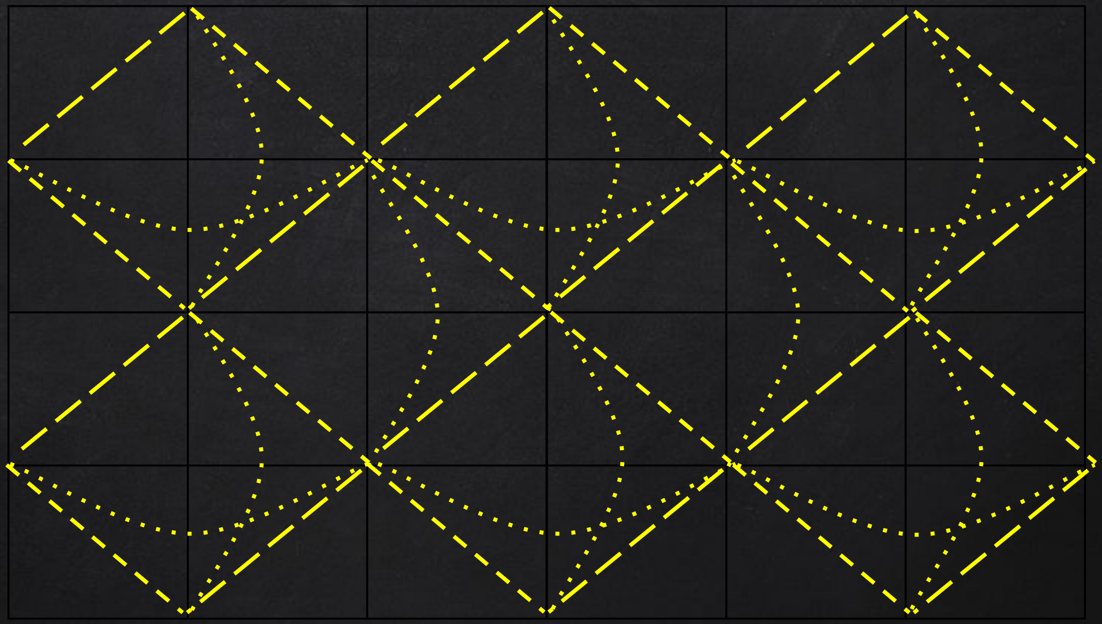
1st NN

2nd NN



1st NN

2nd NN



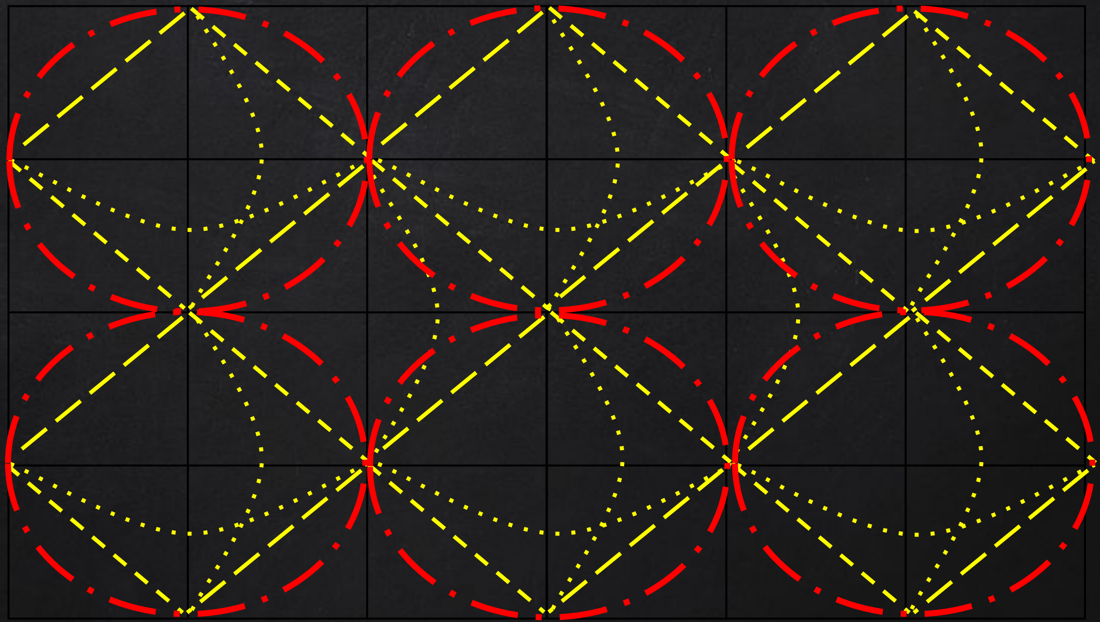
Changing the Square Structure

Hyper-Graph

1st NN: $2S_2$

2nd NN: S_2

Emergent Term: S_4

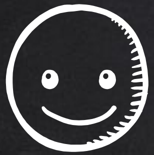


$$2S_2 = \frac{1}{4} ln \cosh 4\beta$$

$$S_2 = \frac{1}{8} ln \cosh 4\beta$$

$$S_4 = \frac{1}{8} ln \cosh 4\beta - \frac{1}{2} ln \cosh 2\beta$$

$$2S_2 > S_2 > S_4$$



$$2S_2 = \frac{1}{4} ln \cosh 4\beta$$

$$S_2 = \frac{1}{8} ln \cosh 4\beta$$

$$S_4 = \frac{1}{8} ln \cosh 4\beta - \frac{1}{2} ln \cosh 2\beta$$

Saving The Square Structure

1st NN

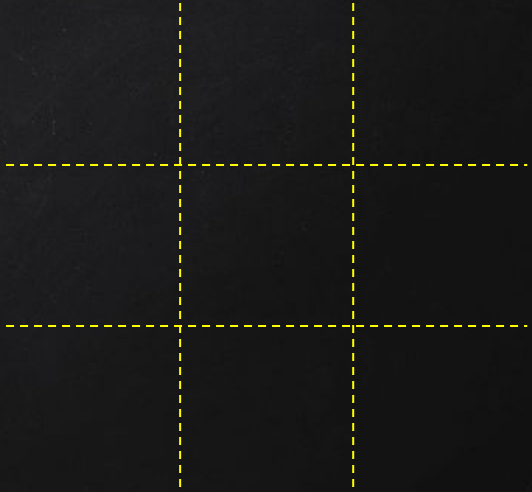
"Approx. RG Group"

β

$\frac{1}{4} ln \cosh 4\beta$

J Isomorphism J'

J is #isomorphic to J '



Saving The Square Structure

1st NN

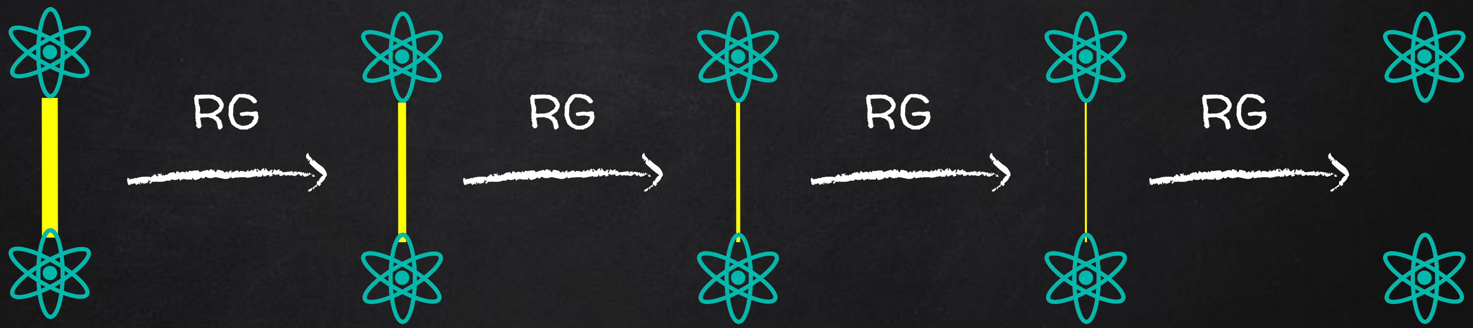
"Approx. RG Group"

$$\beta \longrightarrow \frac{1}{4} ln \cosh 4\beta \longrightarrow \frac{1}{4} ln \cosh 4(\frac{1}{4} ln \cosh 4\beta)$$

Saving The Square Structure

1st NN

"Approx. RG Group"



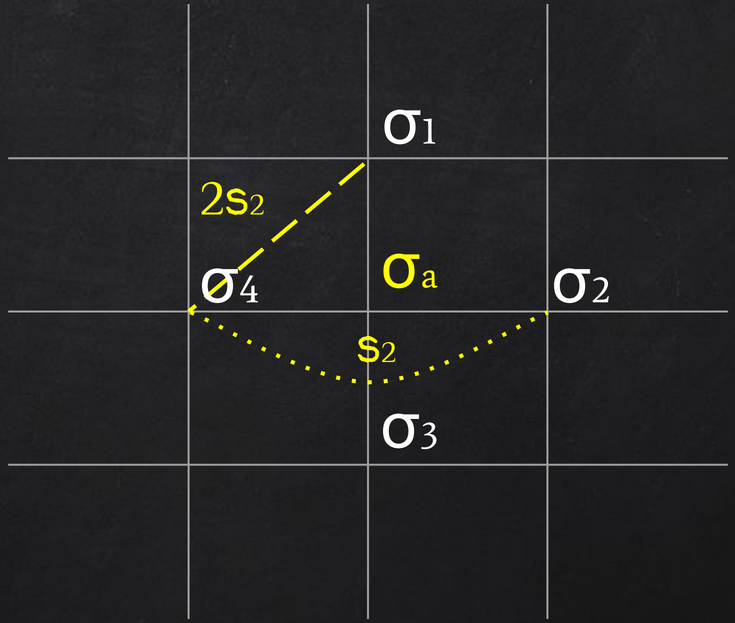
Bad Decimation Transformation!

$$\beta \longrightarrow \frac{1}{4} ln \cosh 4\beta$$

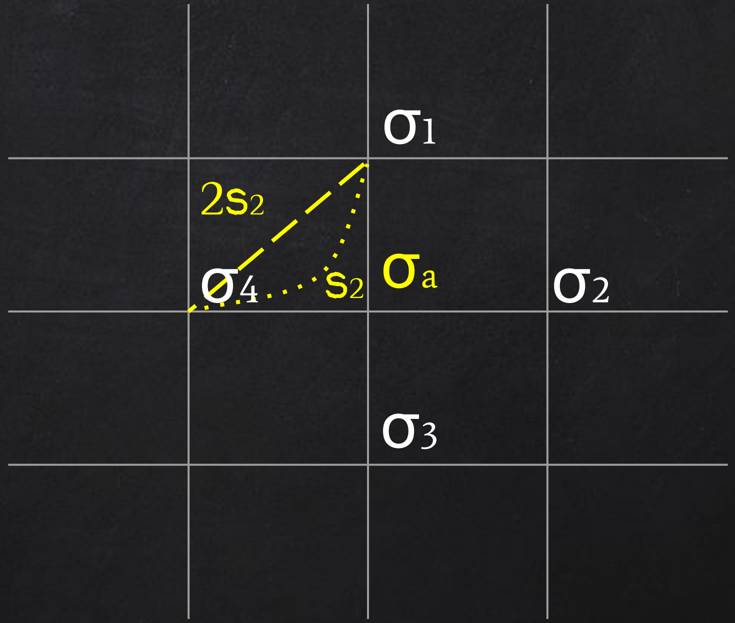
A Better Transformation!

$$S_4 = \frac{1}{8} ln \cosh 4\beta - \frac{1}{2} ln \cosh 2\beta$$

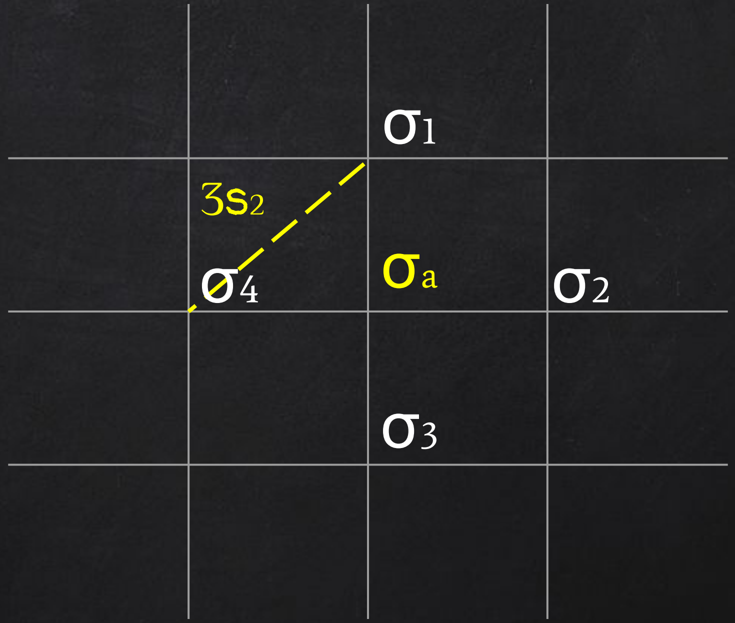
A Better Transformation!



A Better Transformation!



A Better Transformation!



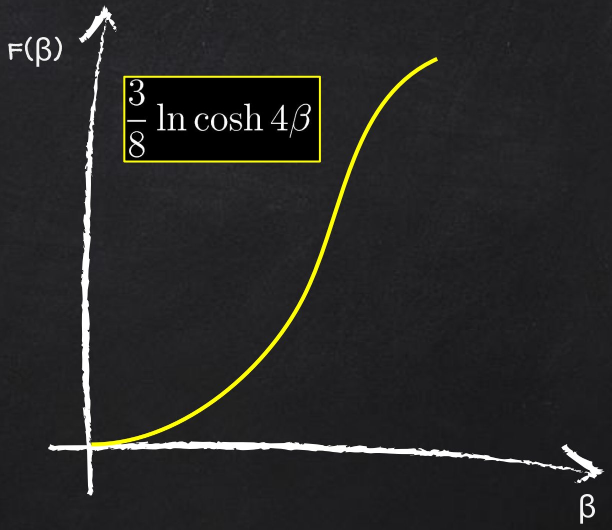
A Better Transformation!

$$\beta \longrightarrow \frac{1}{4} ln \cosh 4\beta$$

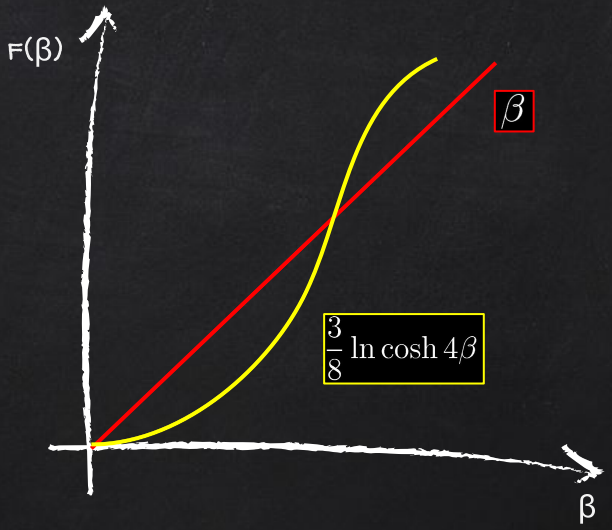
$$\beta \longrightarrow \frac{3}{8} \ln \cosh 4\beta$$

$\beta \longrightarrow \frac{3}{8} \ln \cosh 4\beta$

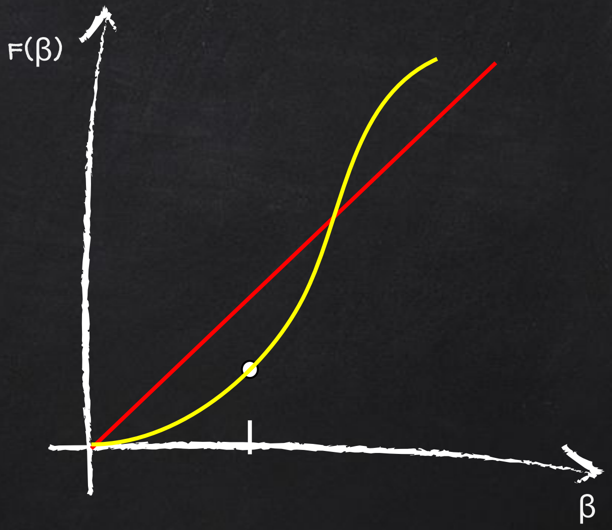
$$\beta \longrightarrow \frac{3}{8} \ln \cosh 4\beta$$



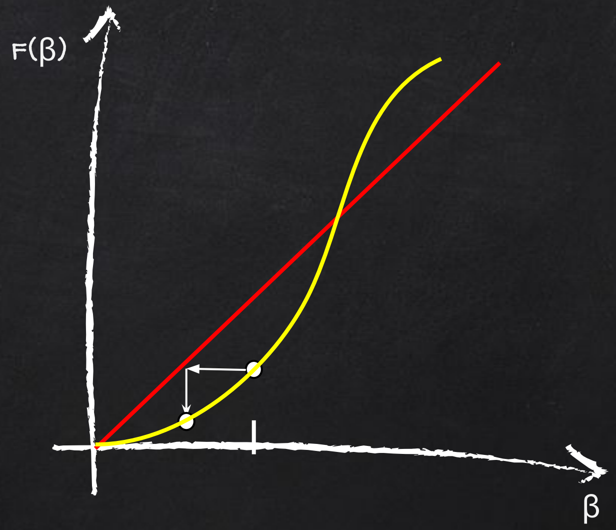
$$\beta \longrightarrow \frac{3}{8}\ln\cosh 4\beta$$



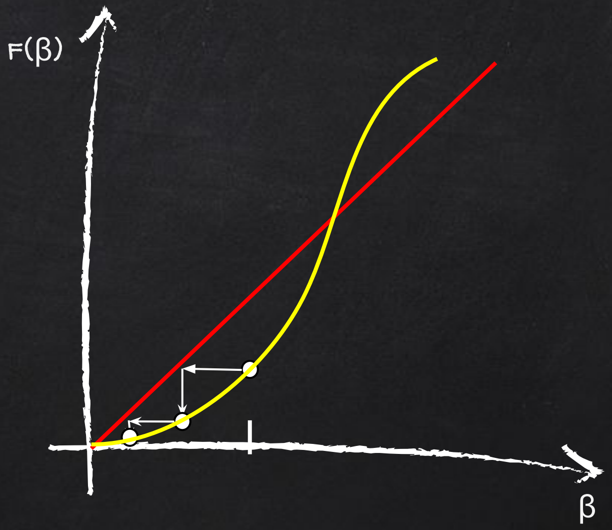
$$\beta \longrightarrow \frac{3}{8}\ln\cosh 4\beta$$



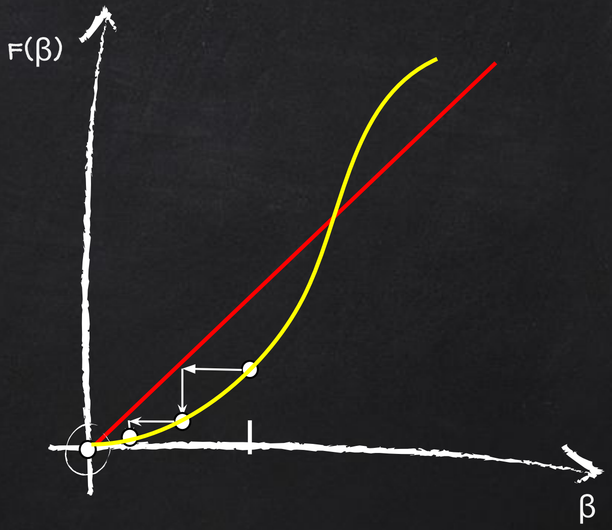
$$\beta \longrightarrow \frac{3}{8}\ln\cosh 4\beta$$



$$\beta \longrightarrow \frac{3}{8} \ln \cosh 4\beta$$

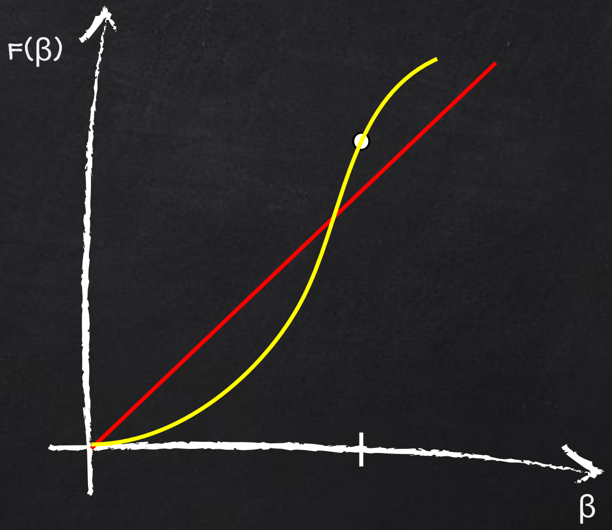


$$\beta \longrightarrow \frac{3}{8} \ln \cosh 4\beta$$



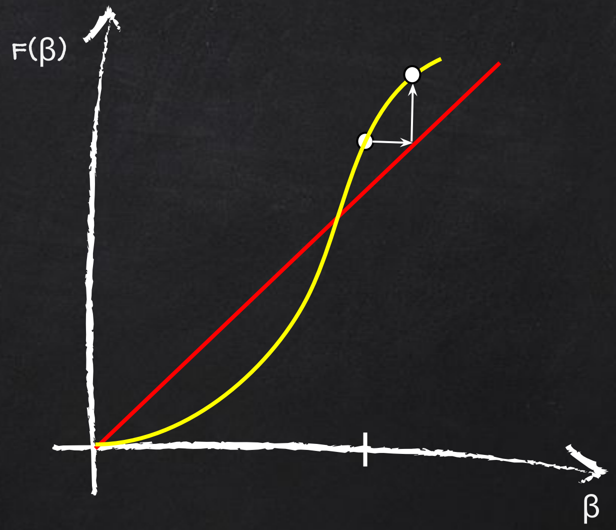
$$\beta \longrightarrow \frac{3}{8} \ln \cosh 4\beta$$

$\frac{1}{\beta}$

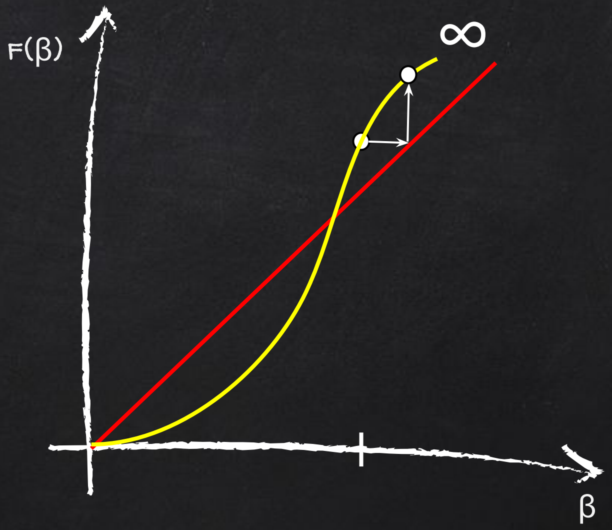


Low T

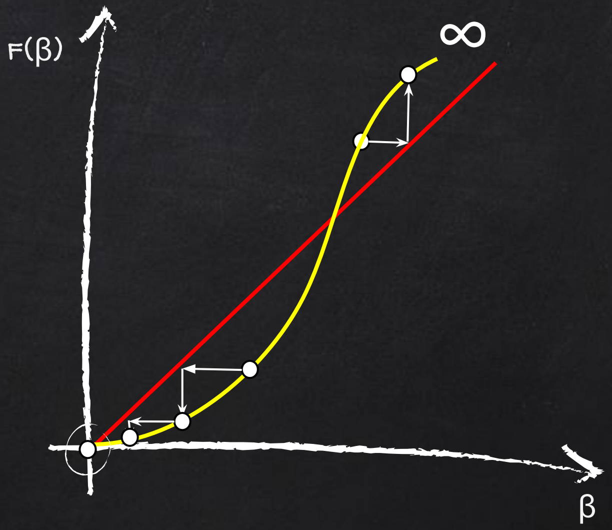
$$\beta \longrightarrow \frac{3}{8}\ln\cosh 4\beta$$



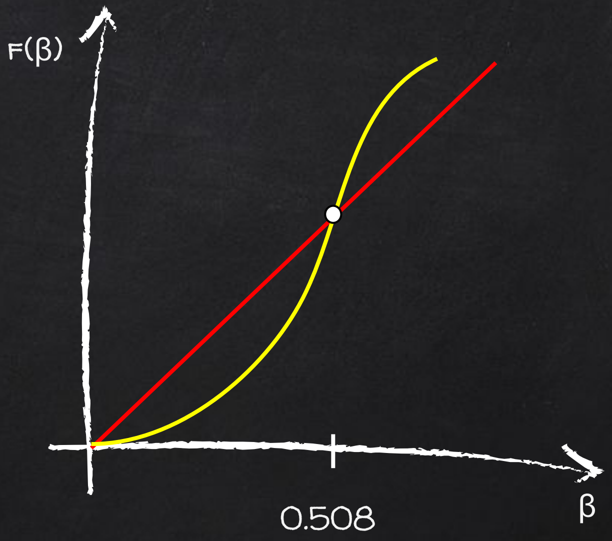
$$\beta \longrightarrow \frac{3}{8} \ln \cosh 4\beta$$



$$\beta \longrightarrow \frac{3}{8} \ln \cosh 4\beta$$

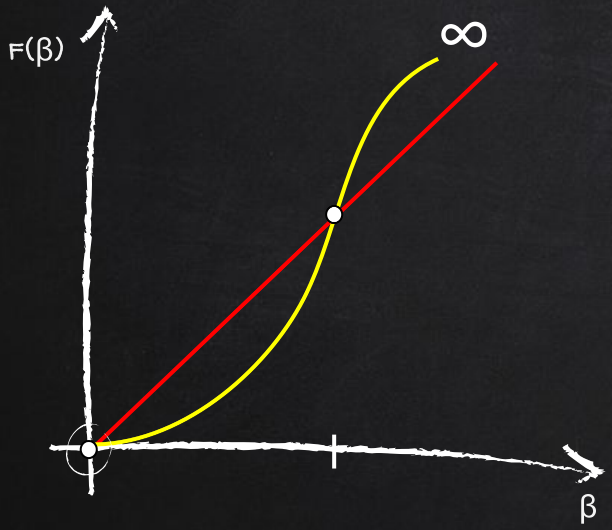


$$\beta \longrightarrow \frac{3}{8}\ln\cosh 4\beta$$



Self-similarity!

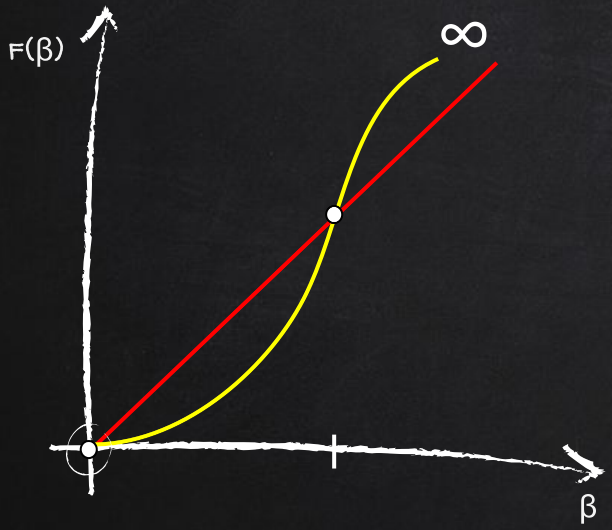
$$\beta \longrightarrow \frac{3}{8}\ln\cosh 4\beta$$

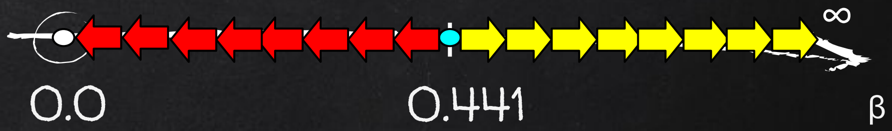


$$S_4 = \frac{1}{8} ln \cosh 4\beta - \frac{1}{2} ln \cosh 2\beta$$

$$\beta = 0.508, S_4 = 0.05$$

$$\beta \longrightarrow \frac{3}{8} \ln \cosh 4\beta$$



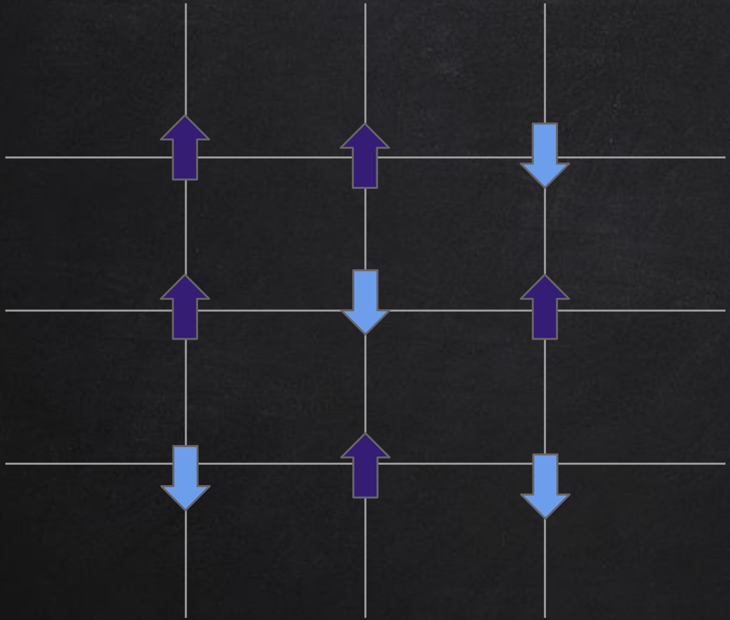


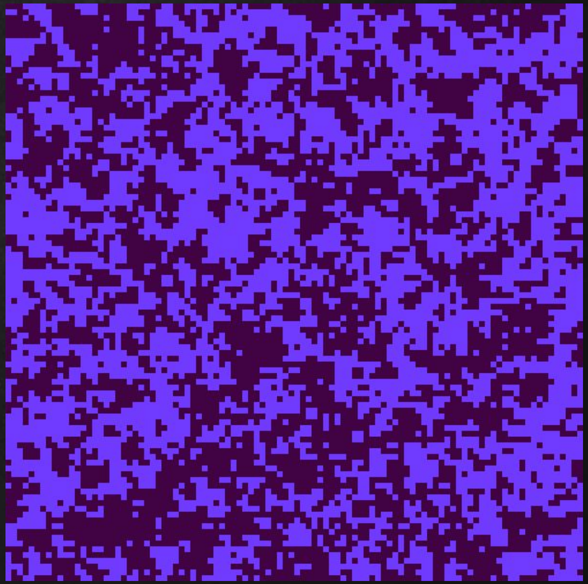
Exact Solution for 2D Ising Model

Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition

Phys. Rev. 65, 117 – Published 1 February 1944

Lars Onsager





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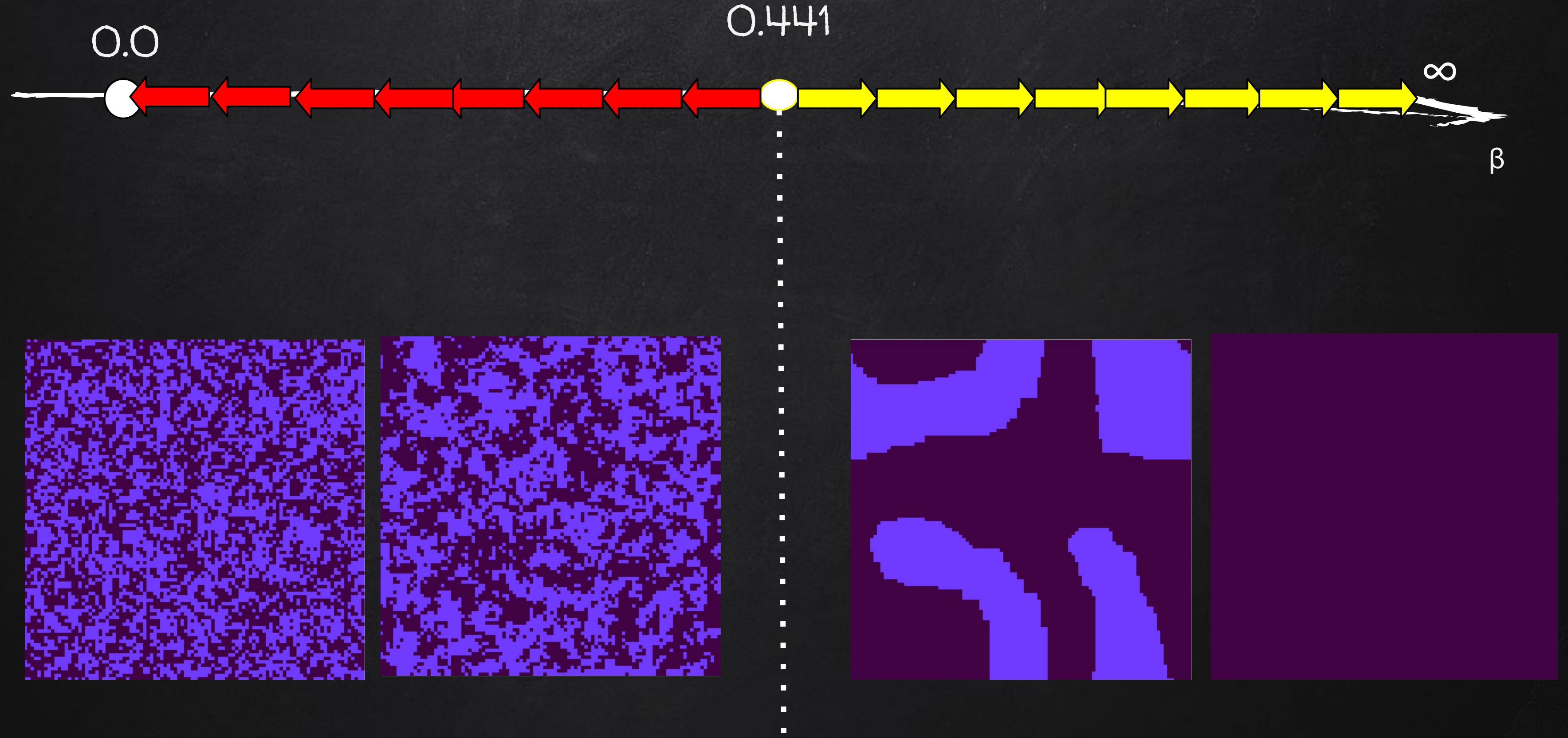

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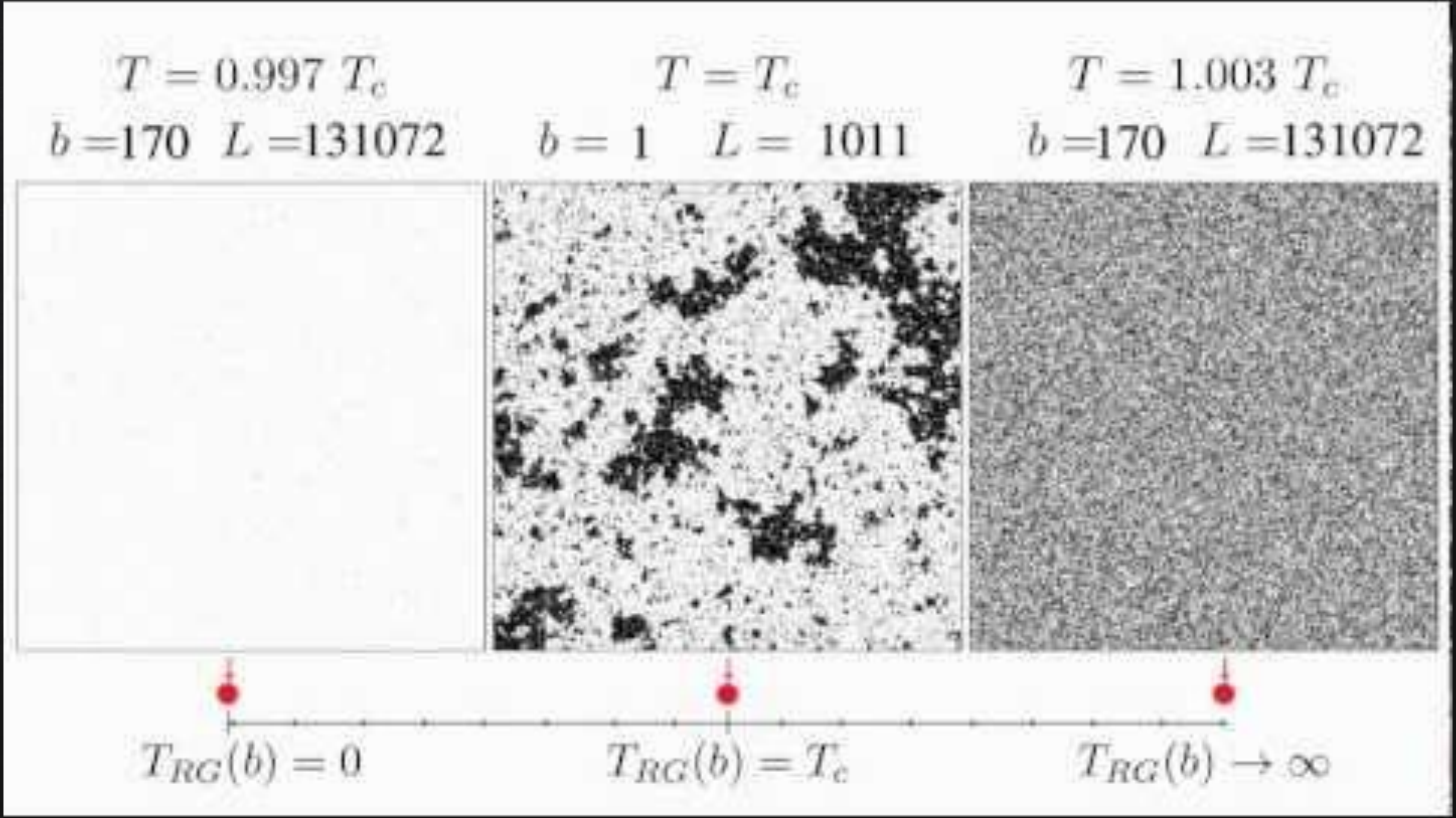
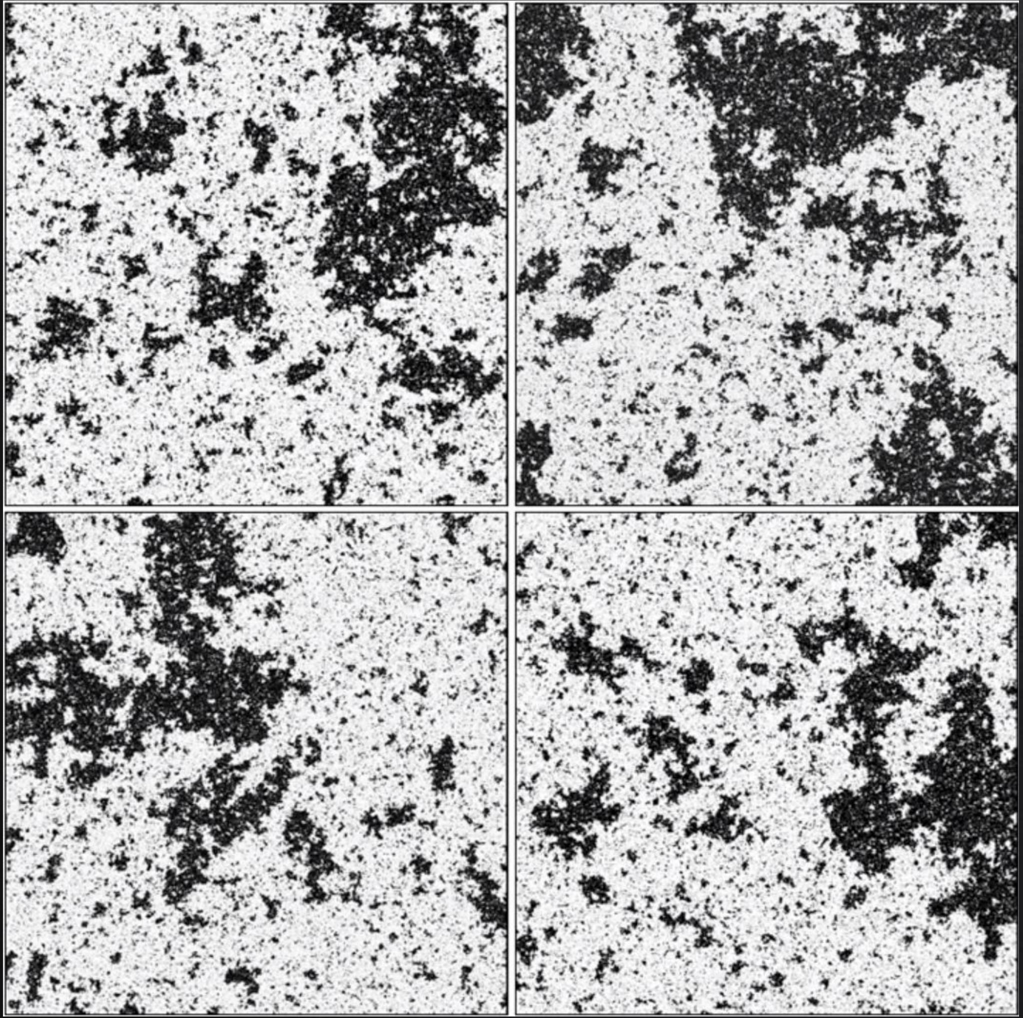


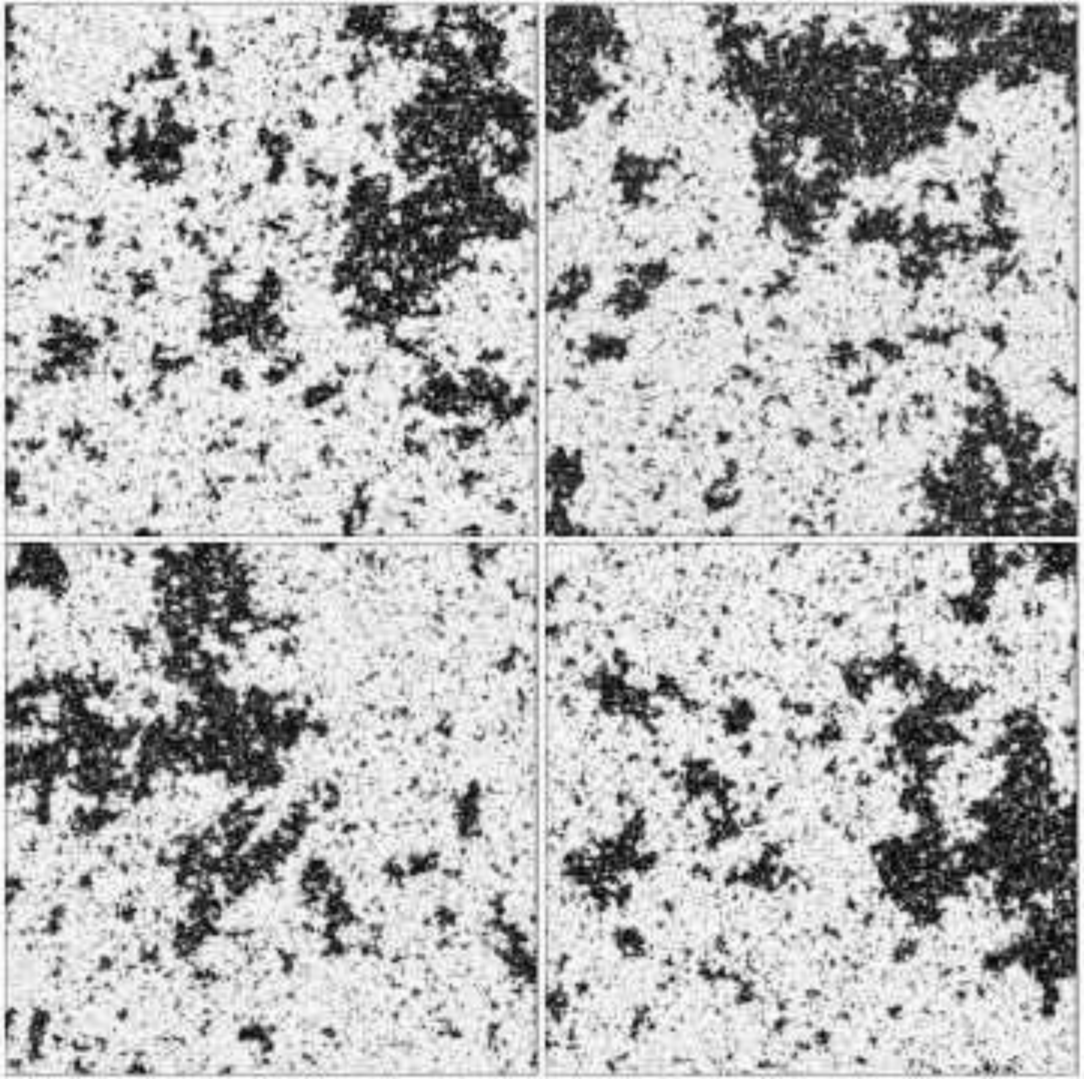


$L = 2^{17}$

$L = 2^{11}$

KINETICALLYCONSTRAINED.COM

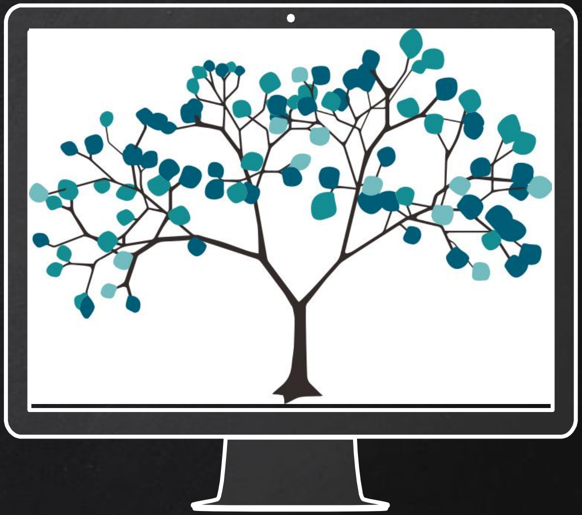
HTTPS://YOUTU.BE/FI-G2ET97X8



Thanks!

Abbas K. Rizi

ABBAS.SITPOR.ORG



✘ bit.ly/SFIrenorm